Central-Upwind Scheme on Triangular Grids for the Saint-Venant System of Shallow Water Equations

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Abstract. We consider a novel second-order central-upwind scheme for the Saint-Venant system of shallow water equations on triangular grids which was originally introduced in [3] . Here, in several numerical experiments we demonstrate accuracy, high resolution and robustness of the proposed method.

Saint-Venant System of Shallow Water Equations

We consider the two-dimensional (2-D) Saint-Venant system of shallow water equations:

$$h_t + (hu)_x + (hv)_y = 0,$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = -ghB_x,$$

$$(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y = -ghB_y, \tag{1}$$

where the function $B(x,y)$ represents the bottom elevation, h is the fluid depth above the bottom, $(u,v)^T$ is the velocity vector, and g is the gravitational constant. This system is widely used in many scientific and engineering applications related to modeling of water flows in rivers, lakes and coastal areas. The development of robust and accurate numerical methods for the computation of its solutions is an important and challenging problem. One of the difficulties encountered is the fact that system (1) admits nonsmooth solutions: shocks, rarefaction waves and, when the bottom topography function B is discontinuous, contact discontinuities. In the latter case, the solution may not be unique, which makes the design of robust numerical methods more challenging even in the one-dimensional (1-D) case (see, e.g., [1]).

A good numerical method for (1) should have two major properties, which are crucial for its stability:

 (i) The method should be well-balanced, that is, it should exactly preserve the stationary steady-state solutions $h + B \equiv \text{const}$, $u \equiv v \equiv 0$ (lake at rest states).

(ii) The method should be positivity preserving, that is, the water depth h should be nonnegative at all times.

In the past decade, a number of well-balanced [2, 5, 7, 17, 8, 10, 11, 12, 13, 14, 15, 16, 18, 19] and positivity preserving [2, 8, 10, 14] schemes for (1) have been proposed, *but only few of them satisfy both major properties (i) and (ii).* For example, see [8], where well-balanced and positivity preserving central-upwind schemes have been introduced. However, the schemes presented in [8] do not simultaneously satisfy (i) and (ii) over the entire computational domain. In a recent work [10], a new second-order central-upwind scheme, which is well-balanced and positivity preserving at the same time, has been proposed.
In both [8] and [10], the central-upwind schemes for the 2-D system (1) are developed for Cartesian grids. Many

real world engineering applications require the use of triangular meshes due to the complicated structure of the computational domains of the problems being investigated. A well-balanced central-upwind scheme on triangular grids has been recently developed in [4], where the presented "triangular" scheme is a (nonconservative) modification of the "triangular" central-upwind scheme from [9] with a special quadrature for the source average over arbitrary triangular cells. The method in [4] is not claimed to be positivity preserving, and is expected to fail on dry states ($h \approx 0$).

In our work [3], we derive the new simple and efficient second order well-balanced positivity preserving central-upwind scheme on triangular grids. Like the central-upwind scheme from [4], our scheme is well-balanced. However, the new quadrature for the discretization of the geometric source introduced in our scheme, is much simpler than the one proposed in [4]. Additionally, unlike the scheme from [4], the central-upwind scheme proposed in our work is positivity preserving. The latter property is achieved by replacing the (possibly discontinuous) bottom topography function B with its continuous piecewise linear approximation and adjusting the piecewise linear reconstruction for w according to the piecewise linear approximation of B. Here, we illustrate the property of our scheme in several numerical examples.

Example 1 — Small Perturbation of a Stationary Steady-State Solution

We first solve the initial value problem (IVP), which is a modification of the benchmark proposed in [11]. The computational domain is $[0,2] \times [0,1]$ and the bottom consists of an elliptical shaped hump:

$$B(x,y) = 0.8 \exp(-5(x-0.9)^2 - 50(y-0.5)^2). \tag{2}$$

Initially, the water is at rest and its surface is flat everywhere except for $0.05 < x < 0.15$:

$$w(x,y,0) = \begin{cases} 1+\varepsilon, & 0.05 < x < 0.15, \\ 1, & \text{otherwise}, \end{cases} \qquad u(x,y,0) \equiv v(x,y,0) \equiv 0, \tag{3}$$

where ε is the perturbation height. We have used zero-order extrapolation at the right and the left boundaries, while the boundary conditions in the y-direction are periodic.

We take a small perturbation height $\varepsilon = 10^{-2}$. Figure 1 displays the right-going disturbance as it propagates past the hump. The water surface, $w(x,y,t)$, computed on the mesh using $2 \times 400 \times 400$ triangles, is presented at times $t = 0.74$ (left) and 1.48 (right). One can observe the high resolution of complex small features of the flow (compare with [4, 8, 11]) and no "parasitic" waves are present.
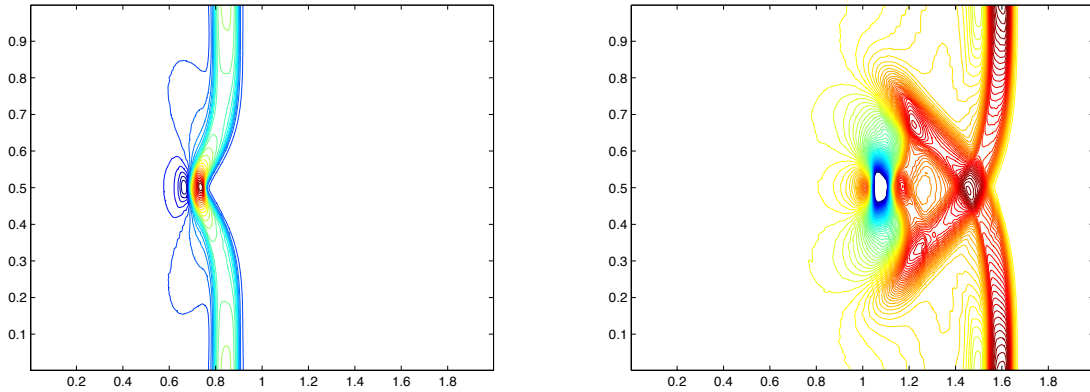

FIGURE 1. w component of the solution of the IVP with $\varepsilon = 10^{-2}$, computed by the well-balanced central-upwind schemes.

Example 2 — Flow in Converging-Diverging Channel

In the second example we study water flow in open converging-diverging channel of length 3 with symmetric constrictions of length 1 at the center [6, 4, 3]. The exact geometry of each channel is determined by its breadth, which is equal to $2y_b(x)$, where

$$y_b(x) = \begin{cases} 0.5 - 0.25\cos^2(\pi(x-1.5)), & |x-1.5| \le 0.5, \\ 0.5, & \text{otherwise}, \end{cases}$$

The computational domain is $[0,3] \times [-y_b(x), y_b(x)]$, see Figure 2. For this test we consider the following initial data:

$$w(x,y,0) = \max\left\{1, B(x,y)\right\}, \quad u(x,y,0) = Fr, \quad v(x,y,0) = 0 \tag{4}$$

where we take Froude number $Fr = 2$. In Fig. (3) we illustrate the performance of our scheme for flow in converging-diverging channel with bottom topography
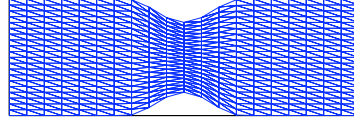
$$B(x,y) = 0 \tag{5}$$

FIGURE 2. Flow in converging-diverging channel: unstructured triangular mesh used for the computations.
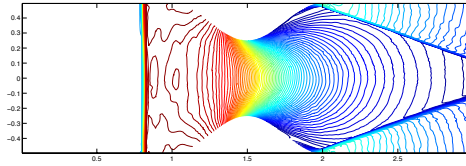

FIGURE 3. Flow in converging-diverging channel: steady-state solution (w) for $B = 0$ on $2 \times 200 \times 200$ and Froude number $Fr = 2$

One can observe the accuracy and high resolution of the method.

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