Translation lengths in $Out(F_n)$

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Abstract

We prove that all elements of infinite order in $Out(F_n)$ have positive translation lengths; moreover, they are bounded away from zero. As a consequence we get a new proof that solvable subgroups of $Out(F_n)$ are finitely generated and virtually abelian.

1 Introduction

In this paper we will study the translation lengths of outer automorphisms of a free group. Following [GS91] we define the translation length $\tau_{X,G}(g)$ of $g \in \Gamma$ to be

$$\lim_{n\to\infty} \frac{\|g^n\|}{n}$$

where Γ is a group with finite generating set X, and $\|g\|$ denotes the length of g in the word metric on Γ associated to X.

Farb, Lubotzky and Minsky proved that Dehn twists (more generally, all elements of infinite order) in $Mod(\Sigma_g)$ have positive translation length ([FLM]). We prove

Theorem 1.1. *Every infinite order element $\mathcal{O} \in Out(F_n)$ has positive translation length. Furthermore, there exists a positive constant ε_n such that $\tau(\mathcal{O}) \geq \varepsilon_n, \quad \forall \mathcal{O} \in Out(F_n)$.*

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Once more we can see the strong analogy between mapping class group of a surface, $Mod(\Sigma_g)$, and outer automorphism group of a free group, $Out(F_n)$.

To prove their theorem, Farb, Lubotzky and Minsky found a way to measure how much a Dehn twist is 'twisted' by looking at simple closed curves and their intersection number. Such an approach cannot work in the case of $Out(F_n)$ as we do not have an analogue of the intersection number.

As a consequence of our main result we have

Corollary 1.2. *Every abelian subgroup of $Out(F_n)$ is finitely generated.*

Corollary 1.3. *Every solvable subgroup of $Out(F_n)$ is finitely generated and virtually abelian.*

Corollary 1.3 was proved in [BFH99a], but Theorem 1.1 offers an alternative proof. Proofs of corollaries 1.2 and 1.3 for Artin groups can be found in [Bes99, 4.2, 4.4]. Artin groups and $Out(F_n)$ share the properties crucial to the aforementioned proofs. In particular they are virtually torsion free, their virtual cohomological dimension is finite, and the translation length restricted to a torsion free abelian subgroup is a norm on that subgroup. The last fact for $Out(F_n)$ follows from the Theorem 1.1.

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2 Translation lengths

From the definition of translation length we can see that it depends on the choice of generating set for a group Γ. We will omit the reference to the generating set, since it will be clear which one we are using.

We list some properties of translation lengths which can be found in [GS91].

Proposition 2.1. *Let X be a generating set for a group Γ.*

1. $0 \leq \tau(g) \leq \|g\|$
2. *For all $x, g \in G$, $\tau(xgx^{-1}) = \tau(g)$.*
3. $\tau(g^n) = n \cdot \tau(g)\ \forall n \in \mathbb{N}$.

Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of generators of a free group F_n. Let Y be the set of generators for $Aut(F_n)$ consisting of:

1. permutations ($x_i \mapsto x_j$, $x_j \mapsto x_i$, $x_k \mapsto x_k$ for all $k \neq i, j$),
2. inversions ($x_i \mapsto x_i^{-1}$, $x_j \mapsto x_j$ for all $j \neq i$),
3. Nielsen twists ($x_i \mapsto x_i x_j$, $x_k \mapsto x_k$ for all $k \neq i$).

Let $\tilde{Y}$ denote the generating set for $Out(F_n)$ consisting of equivalence classes of elements of Y.

Our goal is to prove that every element of infinite order in $Out(F_n)$ has positive translation length. Since $Aut(F_n)$ embeds into $Out(F_{n+1})$, it will follow that every infinite order element of $Aut(F_n)$ has positive translation length.

We will need the following definition for our proof:

Definition 2.2. Define a map $\alpha : F_n \to \mathbb{N}$ by

$$\alpha(w) = \text{the largest } p \geq 0 \text{ such that for some nontrivial reduced word } u \text{ the word } u^p \text{ is a subword of } w,$$

where elements of F_n are regarded as reduced words in the generators and their inverses. We also define

$$\tilde{\alpha}([w]) = \max\{\alpha(u) : u \text{ is a cyclically reduced conjugate of } w\}$$

for the conjugacy class, $[w]$, of w.

Example 2.3.

$$\alpha(1) = 0$$

$$\alpha(ab^{-7}c) = 7$$

Lemma 2.4. *There exists a constant $C > 0$ such that for any $\tilde{g} \in \tilde{Y}$ and any cyclically reduced word $w \in F_n$ we have*

$$\tilde{\alpha}(\tilde{g}([w])) \leq \tilde{\alpha}([w]) + C\,.$$

Proof. Let $w \in F_n$ be a cyclically reduced element of length n with $\alpha(w) = p$. Write $w = A\, u^p\, B$, for some $u \in F_n$. Consider

$$g(w) = [[g(A)]]\, [[g(\tilde{w})^p]]\, [[g(B)]]\,,$$

where $[[x]]$ denotes the reduced word obtained from x. By the *Bounded Cancellation Lemma* ([Coo87]) there is a constant $C(g)$ such that at most $C(g)$ cancellations occur after concatenation of the words $[[g(A)]]$ and $[[g(\tilde{w})^p]]$. Hence p can decrease by at most $2\,C(g)$ (cancellations may occur at the beginning and at the end of $[[g(\tilde{w})^p]]$). Let $C_g = 2\,\max\{C(g), C(g^{-1})\}$. We now have

$$\begin{aligned}
&\alpha([[g(w)]]) \geq \alpha(w) - C_g \\
&\alpha(w) = \alpha(g^{-1}(g(w))) \geq \alpha([[g(w)]]) - C_g \\
&\alpha([[g(w)]]) \leq \alpha(w) + C_g.
\end{aligned}$$

If we take $C = \max\{C_g : g \in Y\}$, our claim is proved for elements of Y.

Let $\tilde{g} \in \tilde{Y}$ and let g be a representative for the equivalence class $\tilde{g}$. The argument in this case differs from the above argument in that after applying g to w, p can decrease by at most $3\,C(g)$ (it may happen that $g(w)$ is not cyclically reduced and we can get cancellation at the ends of $g(w)$). We now proceed as above.

□

Example 2.5. We illustrate the idea of the proof of Theorem 1.1 with an example of a Nielsen twist. Let g be a Nielsen twist which sends x_2 to x_2x_1 and fixes all other generators of F_n.

$$\alpha(g^k(x_2)) = \alpha(x_2x_1^k) = k\,.$$

Write $g^k = g_1 \cdots g_m$ with $g_i \in Y$ and $m = \|g^k\|$. By Lemma 2.4, we have that

$$\begin{aligned}
k &= \alpha(g^k(x_2)) \leq \alpha(x_2) + m\,C = m\,C + 1\,, \\
\tau(g) &= \lim_{k\to\infty} \frac{\|g^k\|}{k} \geq \lim_{k\to\infty} \frac{k-1}{k\,C} = \frac{1}{C} > 0\,.
\end{aligned}$$

So g has positive translation length.

We give a short list of definitions which will be used throughout the rest of the paper, but we suggest that the reader look at [BFH99b].

Every element $\mathcal{O} \in Out(F_n)$ can be represented by a homotopy equivalence $f\colon G \to G$ of a graph G whose fundamental group is identified with F_n. A map $\sigma\colon J \to G$ (J is an interval) is called a *path* if it is either locally injective or a constant map (we also require that the endpoints of σ are at vertices). Every map $\sigma\colon J \to G$ is homotopic (relative endpoints) to a path $[[\sigma]]$.

If $\sigma = \sigma_1 \ldots \sigma_l$ is a decomposition of a path or a circuit σ into nontrivial subpaths we say that it is a *k-splitting* if

$$f^k(\sigma) = [[f^k(\sigma_1)]] \ldots [[f^k(\sigma_l)]]$$

is a decomposition into subpaths and is a *splitting* if it is a k-splitting for all $k > 0$.

We say that a nontrivial path $\sigma \in G$ is a *Nielsen path* for $f : G \to G$ if $[[f(\sigma)]] = \sigma$. The Nielsen path σ is *indivisible* if it cannot be written as a concatenation of nontrivial Nielsen paths.

Let $= G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_K = G$ be a filtration of G by f-invariant subgraphs, and let $H_i = \overline{G_i \backslash G_{i-1}}$. Suppose H_i is a single edge E_i and $f(E_i) = E_i v^l$ for some closed indivisible Nielsen path $v \subset G_{i-1}$ and some $l > 0$. The *exceptional paths* are paths of the form $E_i v^k \overline{E_j}$ or $E_i \overline{v}^k \overline{E_j}$, where $k \geq 0$, $j \leq i$ and $f(E_j) = E_j v^m$, for $m > 0$.

We remind the reader that every element of $Out(F_n)$ of infinite order has either exponential or polynomial growth ([BH92]). A polynomially growing outer automorphism $\mathcal{O} \in Out(F_n)$ is unipotent if its action in $H_1(F_n;\mathbb{Z})$ is unipotent (UPG automorphism).

The following Theorem can be found in [BFH00](page 564).

Theorem 2.6. *Suppose that $\mathcal{O} \in Out(F_n)$ is a UPG automorphism. Then there is a topological representative $f\colon G \to G$ of $\mathcal{O}$ with the following properties:*

1. *Each G_i is the union of G_{i-1} and a single edge E_i satisfying $f(E_i) = E_i \cdot u_i$ for some closed path u_i that crosses only edges in G_{i-1} ($\cdot$ indicates that the decomposition in question is a splitting).*
2. *If σ is any path with endpoints at vertices, then there exists $M = M(\sigma)$ so that for each $m \geq M$, $[[f^m(\sigma)]]$ splits into subpaths that are either single edges or exceptional subpaths.*

□

We need to modify a definition of our map α to the new setting.

Definition 2.7. For a path γ in a graph G let

$$\alpha(\gamma) = \text{the largest } p \geq 0 \text{ such that for some nontrivial path } \sigma \text{ the path } \sigma^p \text{ is a subpath of } \gamma,$$

The map $\tilde{\alpha}$ is defined on circuits in the exactly same way.

The following example demonstrates the difference between these two seemingly identical maps.

Example 2.8. Let $\gamma = abca$.
If γ is a path then $\alpha(\gamma) = 1$,
but if it is a circuit, then $\tilde{\alpha}(\gamma) = 2$.

Lemma 2.9. *Let $\mathcal{O} \in Out(F_n)$ be a UPG automorphism of infinite order and let $f\colon G \to G$ be its topological representative as in Theorem 2.6. For every path γ in G for which $[[f(\gamma)]] \neq \gamma$ there exists $a \in \mathbb{Z}$ such that*

$$\alpha([[f^k(\gamma)]]) \geq k + a\,.$$

Proof. We prove our claim by induction on the (minimal) index, m, of the filtration element that contains a path γ.

If $\gamma \subset G_1$ there is nothing to be proved since G_1 contains only one edge E_1 which is fixed by f.

Suppose the claim is true for the subpaths contained in G_{m-1} that satisfy our hypothesis, and let γ be a path in G_m for which $[[f(\gamma)]] \neq \gamma$. By Theorem 2.6 for every $m \geq M(\gamma)$, $[[f^m(\gamma)]]$ splits into subpaths that are either single edges or exceptional paths. Denote $[[f^{M(\gamma)}(\gamma)]]$ by $\tilde{\gamma}$, so that $\tilde{\gamma} = \gamma_1 \cdot \ldots \cdot \gamma_p$, where γ_i is either a single edge or an exceptional path.

Assume there is an exceptional path γ_t which is not fixed by f. Without loss of generality we may assume that $\gamma_t = E_i v^r \overline{E_j}$, where $f(E_i) = E_i v^l$ ($l > 0$), $f(E_j) = E_j v^s$ ($s > 0$) and $j \leq i$. Now we have that

$$[[f^k(\gamma_t)]] = E_i v^{k(l-s)+r}\ \overline{E_j}\,,$$

and

$$\alpha([[f^k(\gamma_t)]]) \geq k(l-s) + r, \quad \text{if } l - s > 0\,,$$

$$\alpha([[f^k(\gamma_t)]]) \geq k(s-l) - r, \quad \text{if } l - s < 0\,.$$

Since γ_t is not fixed, l and s cannot be equal. Therefore

$$\alpha([[f^k(\tilde{\gamma})]]) \geq k - r\,,$$

$$\alpha([[f^k(\gamma)]]) = \alpha([[f^{k-M(\gamma)}(\tilde{\gamma})]]) \geq k - M(\gamma) - r\,.$$

If all exceptional paths in $\tilde{\gamma}$ are fixed, there exists an edge $\gamma_t = E_i$ which is not fixed by f. We know that $f(E_i) = E_i \cdot u_i$, where u_i is a closed path contained in G_{m-1}.
If $[[f(u_i)]] = u_i$, our claim is proven since $[[f^k(E_i)]] = E_i u_i^k$ and so

$$\alpha([[f^k(\tilde{\gamma})]]) \geq k\,,$$

$$\alpha([[f^k(\gamma)]]) \geq k - M(\gamma)\,.$$

If $[[f(u_i)]] \neq u_i$, there exists $a \in \mathbb{R}$ such that $\alpha([[f^k(u_i)]]) \geq k + a$. We now have

$$\alpha([[f^k(\gamma)]]) = \alpha([[f^{k-M(\gamma)}(\tilde{\gamma})]]) \geq \alpha([[f^{k-M(\gamma)}(u_i)]]) \geq k - M(\gamma) + a\,.$$

□

Lemma 2.10. *Let $\mathcal{O}$ be a UPG automorphism of F_n of infinite order. There exist a closed path σ in G, and $b \in \mathbb{R}$ such that*

$$\tilde{\alpha}(\mathcal{O}^k(\sigma)) \geq k + b\,.$$

Proof. Let $f\colon G \to G$ be as in Theorem 2.6. Since $\mathcal{O} \neq id$ there is a closed path σ which is not fixed by f. We know that for every $m \geq M(\sigma)$, $[[f^m(\sigma)]] = \sigma_1 \cdot \ldots \cdot \sigma_p$ splits into subpaths that are either single edges or exceptional paths. Denote $[[f^{M(\sigma)}(\sigma)]]$ by $\tilde{\sigma}$, so that $\tilde{\sigma} = \sigma_1 \cdot \ldots \cdot \sigma_p$.

If there is an exceptional path σ_t in this splitting which is not fixed by f, we get

$$\tilde{\alpha}(\mathcal{O}^k(\tilde{\sigma})) \geq k - r$$

as in Lemma 2.9.

If all exceptional paths in $\tilde{\sigma}$ are fixed, there exists an edge $\sigma_t = E_i$ such that $f(E_i) = E_i \cdot u_i$, where u_i is a closed path contained in G_{i-1}. By Lemma 2.9 there exists $a \in \mathbb{R}$ such that

$$\alpha(f^k(E_i)) \geq k + a\,.$$

Hence, in all the above cases, there is $b \in \mathbb{R}$ such that

$$\tilde{\alpha}(\mathcal{O}^k(\tilde{\sigma})) \geq k + b\,.$$

□

3 Proof of Theorem 1.1

We consider the cases of exponentially and polynomially growing outer automorphisms separately.

Case 1. Let $\mathcal{O}$ be an exponentially growing outer automorphism of F_n. There exist $\lambda > 1$ and a cyclically reduced word w such that $\ell(\mathcal{O}^k([w])) \geq \lambda^k \ell([w])$, for all $k \geq 1$, where ℓ denotes the cyclic word length (see [BH92]). Let $\mathcal{O}^k = \tilde{g}_1 \dots \tilde{g_m}$, where $\tilde{g}_i \in \tilde{Y}$ and $m = \|\mathcal{O}^k\|$. It is straightforward to show that for all $\tilde{g} \in \tilde{Y}$ and any cyclically reduced word w we have

$$\ell(\tilde{g}([w])) \leq 2\,\ell([w])$$

Using this inequality we obtain:

$$\lambda^k \ell([w]) \leq \ell(\mathcal{O}^k([w])) \leq 2^m \ell([w])$$

Hence

$$m \geq \frac{\log \lambda^k}{\log 2}$$

which implies

$$\tau(\mathcal{O}) \geq \frac{\log \lambda}{\log 2} > 0$$

Case 2. Let $\mathcal{O}$ be a UPG automorphism. Again let $\mathcal{O}^k = \tilde{g}_1 \dots \tilde{g_m}$, where $\tilde{g}_i \in \tilde{Y}$ and $m = \|\mathcal{O}^k\|$. By Lemma 2.10 there is a closed path σ in G such that

$$\tilde{\alpha}(\mathcal{O}^k(\sigma)) \geq k + b$$

Let $u_j = \tilde{g}_j \dots \tilde{g_m}$. Applying Lemma 2.4 we get

$$\tilde{\alpha}(u_i(\sigma)) \leq \tilde{\alpha}(u_{i+1}(\sigma)) + C$$

which yields

$$k + b \leq \tilde{\alpha}(\mathcal{O}^k(\sigma)) \leq mC + \tilde{\alpha}(\sigma)$$
$$\frac{k + b - \tilde{\alpha}(\sigma)}{C} \leq m\,.$$

We have

$$\tau(\mathcal{O}) \geq \lim_{k\to\infty} \frac{k + b - \tilde{\alpha}(\sigma)}{k\,C} = \frac{1}{C}\,.$$

Case 3. If $\mathcal{O}$ is any polynomially growing outer automorphism, then there exists $s \geq 1$, bounded above by some c_2, ([BFH99b, Definition 3.10, Proposition 3.5]) such that $\mathcal{O}^s$ is a UPG automorphism. Then

$$\tau(\mathcal{O}) = \frac{1}{s}\,\tau(\mathcal{O}^s) \geq \frac{1}{C\,s} > 0\,.$$

In all three cases $\tau(\mathcal{O})$ is bounded away from zero:

Case 1. There is a constant $c_1 > 1$ such that $\lambda \geq c_1$ ([BH92]). Therefore $\tau(\mathcal{O})$ is bounded away from zero.

Case 2. $\tau(\mathcal{O}) \geq \frac{1}{C}$, for a fixed C.

Case 3. Since s is bounded by c_2, we get $\tau(\mathcal{O}) \geq \frac{1}{C\,c_2} > 0$.

This completes the proof. □

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