Non-Smooth Gravity Problem

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Summary

This work deals with reconstruction of non-smooth solutions of the inverse gravimetric problem. The problem is non-unique and requires regularization. Regularization methods commonly used in geophysics reconstruct smooth solutions even though geological structures often have sharp contrasts (discontinuities) in properties. A stabilizing functional that does not penalize sharp features is the total variation of the reconstructed model. This approach permits reconstruction of (non-smooth) density functions that represent blocky geological structures. An adaptive gradient scheme is shown to be effective in solving the regularized inverse problem. A simple example consisting of a model with three homogenous blocks illustrates the behavior of the method.

Introduction

Inversion methods commonly used in geophysics aim to recover a smooth model by including a regularizing term that penalizes sharp changes in properties (e.g., "Occam's" inversion; Constable et al., 1987). This work presents an alternative that can handle blocky geological structures with discontinuities in properties. We apply this technique to the gravimetric inverse problem, which is to determine Earth's density distribution from measurements of gravity (or its derivatives) on the surface or in boreholes. This problem is (intrinsically) non-unique, and its numerical solution is unstable: small variations in the data can cause large variations in the solution. Moreover, gravity data can only be measured at discrete points and are contaminated with noise; hence there are always some deviations of the measured data from the true gravity field.

Different regularization techniques have been developed to stabilize the inverse problem and ensure that the solution constructed numerically is close in some sense to the true solution. There is an extensive literature on this subject (e.g., Tikhonov and Arsenin, 1977; Lavrentiev et al., 1980). Most methods used in geophysical imaging give good results for smooth models; however, they are not able to recover blocky structures or non-smooth density distributions. But blocky structures—faults, mineral bodies, dikes, etc.—with abrupt changes of properties are often the targets of geophysical exploration.

We propose to use the total variation (TV) stabilizing functional to drive the solution to blocky structures. Use of total variation functionals was developed in image reconstruction to restore sharp edges and high contrast images (Rudin et al., 1992); it is also used in electrical tomography (Dobson and Santosa, 1994). We use an adaptive gradient algorithm to minimize the misfit functional that combines the data misfit with the total variation of the solution. Numerical simulations show that an inversion scheme regularized by the total variation functional can reconstruct non-smooth features of the density distribution. This can be a very desirable feature for many inverse problems in geophysical exploration.

The inverse gravimetric problem

The inverse gravimetric problem is to determine the distribution of the density inside the earth from the measurements of the gravity field on the surface. Let a domain Ω be a subsurface domain with a spatially varying density of the medium $\rho(x)$. We consider here 2D problem assuming $x = (x_1, x_2)$. The gravity potential $\phi(x)$ generated by this density distribution satisfies Poisson's equation inside the domain Ω:

$$\Delta\phi = -4\pi\gamma\rho, \quad x \in \Omega. \tag{1}$$

Here γ is the gravitational constant. Using Green's function $G(x', x)$, the solution of (1) at the point x' can be written as

$$\phi(x') = \int_\Omega \rho\,(x)\, G\left(x', x\right) dx. \tag{2}$$

Most gravimeters measure variations in the vertical component of the gravity (force) field; i.e., the vertical derivative of the gravity potential, $\partial\phi/\partial x_2$. The measured function g is related to the density ρ (see (2)) through the integral equation:

$$g(x') = \int_\Omega \rho\,(x)\, \frac{\partial}{\partial x_2} G\left(x', x\right) dx. \tag{3}$$

Equation (3) is a Fredholm integral equation of the first kind (for ρ) and can be written in operator form $K\rho = g$.

To solve the inverse problem, we need to solve the integral equation (3) with known function g (or samples of g). It can be shown that the inverse operator K^{-1} is not continuous which implies that the solution ρ does not depend continuously on the measured data (the inverse problem is unstable as wel as non-unique). We assume

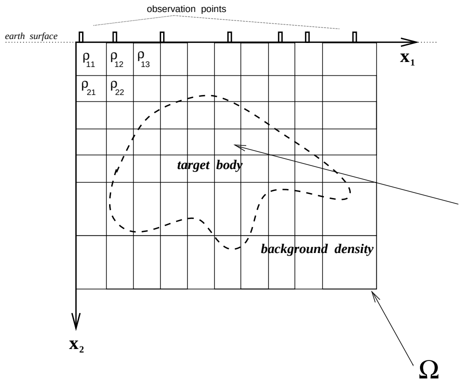

Figure 1: Inverse gravimetric problem is to determine the density of the subsurface from measurements on the surface of the earth. Shown is a discretization of the domain Ω.

that non-uniqueness can be overcome in the standard way by constraining the solution space. We concentrate here on the problem of stable reconstruction of a non-smooth solution.

Smoothing Regularization

A common way to solve inverse problems is by minimization of the misfit functional

$$\min P(\rho, g) = \min_{\rho} \|K\rho - g\|_2^2 . \quad (4)$$

To obtain a stable solution, additional constraints can be introduced for some functional of the solution $J(\rho)$. This leads to a regularized minimization problem:

$$\min P^{\alpha}(\rho, g) = \min \|K\rho - g\|_2^2 + \alpha\, J(\rho), \quad (5)$$

where α is a parameter of regularization. Choice of the functional $J(\rho)$ is very important because it determines what functions can be minimizers of this problem. Some popular quadratic stabilization functionals are the L_2 norm of the solution or of its derivatives (e.g., Tikhonov and Arsenin, 1977; Constable et al., 1987; Bassrei, 1993; Bertete-Aguirre and Xavier, 1994):

$$J_2(\rho) = \|\rho\|_2^2 , \quad \text{or} \quad J_{\nabla}(\rho) = \|\nabla\rho\|_2^2 . \quad (6)$$

These functionals impose restrictions on the smoothness of the solution and are successful when the true solution really satisfies such restrictions. But if the solution is not a smooth function, this type of regularization does not lead to good results (Bertete-Aguirre et al., 1997).

Non-smooth regularization

An alternative to smooth regularization is to use a functional that does not penalize sharp changes in properties (discontinuities). We use the variation of the solution ρ in the domain Ω as the stabilizing functional:

$$\min P_{TV}^{\alpha}(\rho, g) = \min \|K\rho - g\|_2^2 + \alpha\, J_{TV}(\rho). \quad (7)$$

The variation of a function (Giusti, 1984) is a total change of the values of the function in the domain. It sums up the jumps of the function at discontinuity points and its variation in the intervals where the function is continuous. When the function ρ is differentiable, $J_{TV}(\rho)$ is:

$$J_{TV}(\rho) = \int_{\Omega} |\nabla\rho| \; dx, \quad |\nabla\rho| = \sqrt{\left(\frac{\partial\rho}{\partial x_1}\right)^2 + \left(\frac{\partial\rho}{\partial x_2}\right)^2} . \quad (8)$$

We will use this expression understanding that if the function is discontinuous at some point, its variation there is just the size of the jump. For example, if a function is zero on the interval from zero to 0.5, and then jumps and equal to one on the rest of the interval, its variation is one.

To solve (7), we derive the corresponding Euler-Lagrange equations, and then use them to minimize the functional (7) in an adaptive gradient minimization scheme. For the first term in (7), the equation is,

$$(K^*K\rho - K^*g) = 0 \quad (9)$$

The Euler equation for the second term in (7) comes from the following derivation:

$$\delta_{\rho} \int_{\Omega} |\nabla\rho| \; dx = \int_{\Omega} \frac{\nabla\rho \cdot \nabla\delta\rho}{|\nabla\rho|} dx. \quad (10)$$

Integrating by parts recasts the previous integral as

$$\int_{\partial\Omega} \frac{\delta\rho}{|\nabla\rho|} \frac{\partial\rho}{\partial n} \, ds - \int_{\Omega} \delta\rho \left(\nabla \cdot \frac{\nabla\rho}{|\nabla\rho|}\right) dx \quad (11)$$

The optimality condition is satisfied if

$$\nabla \cdot \frac{\nabla\rho}{|\nabla\rho|} = 0 \;\text{ in }\; \Omega, \quad \frac{\partial\rho}{\partial n} = 0 \;\text{ on }\; \partial\Omega. \quad (12)$$

The second of these equations is satisfied if we deal with an isolated mass anomaly, which is a standard assumption for geophysical inverse problems. The lefthand side of the first equation gives us the gradient of the TV functional.

To deal with the term $1/|\nabla\rho|$ in the equation (12), we introduce a small positive parameter β: $|\nabla\rho| \cong \sqrt{|\nabla\rho|^2 + \beta}$ Such an approximation is continuous (Vogel and Oman, 1996).

The resulting Euler-Lagrange equation is

$$(K^*K\rho - K^*g) + \alpha\nabla \cdot \frac{\nabla\rho}{\sqrt{|\nabla\rho|^2 + \beta}} = 0 \quad (13)$$

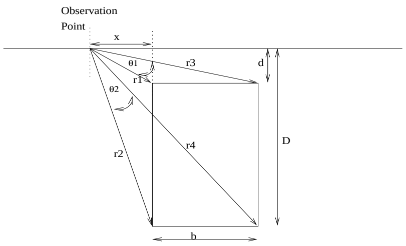

Figure 2: To the calculation of a prism gravity effect.

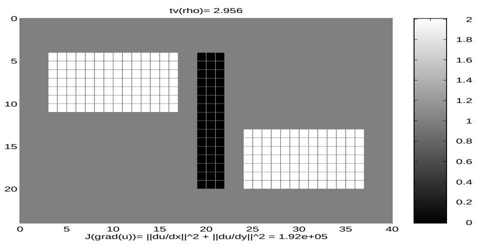

Figure 3: The model density distribution.

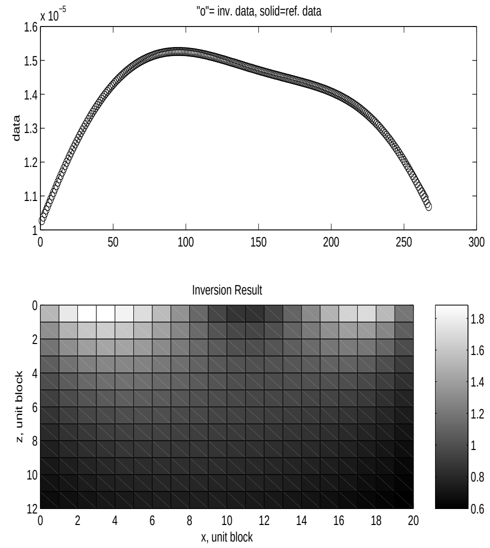

Figure 4: Solution for a smoothing regularization scheme.

Different approaches have been used to solve this equation (Rudin, 1992; Chan et al., 1996; Vogel, 1996; Bertete-Aguirre, 1997). In this work, we discretize the forward operator K and the gradient of the TV functional, and then use a gradient minimization to find the solution of the regularized finite-dimensional problem.

To discretize the problem, we represent Ω as a union of small rectangular blocks: $\Omega = \{\Omega_{ij}, i = 1...nx_1, j = 1...nx_2\}$. The adopted discretization uses rectangular blocks with their principal axes oriented in the same direction as the gravitational attraction vector and its normal defined at the reference surface. The function $\rho(x)$, $x = (x_1, x_2)$, is a piecewise constant function:

$$\rho(x) = \sum_{i=1}^{nx_1} \sum_{j=1}^{nx_2} \rho_{ij} \chi_{ij}(x) \ , \quad x \in \Omega, \tag{14}$$

$$\text{where} \quad \chi_{ij}(x) = \begin{cases} 1, \ x \in \Omega_{ij} \\ 0, \ otherwise. \end{cases} \tag{15}$$

To discretize the operator K, we use an analytic solution of the forward gravity problem for a rectangular prism (Telford et al., 1990):

$$g = 2\gamma \left[D\theta_2 - d\theta_1 - x \log\left(\frac{r_1 r_4}{r_2 r_3}\right) + b \log\left(\frac{r_4}{r_3}\right)\right] \rho. \tag{16}$$

Figure 2 shows the parameters in (16).

The solution ρ is constructed as a minimizer of the functional (7) in an iterative gradient minimization scheme. On each iteration step k, we calculate the gradient of the parametric functional (7) at the point $\rho^{(k)}$, and update the solution

$$\rho^{(k+1)} = \rho^{(k)} + \tau \frac{\partial P^{\alpha}_{TV}(\rho, g)}{\partial \rho}. \tag{17}$$

Here τ is the step in the direction of the maximal descend which minimizes the value of the parametric functional,

$$P^{\alpha}_{TV}(\rho^{(k+1)}, g) \to \min_{\tau}. \tag{18}$$

We use linear search to solve this minimization problem.

Analysis of the numerical results

For numerical simulation we used a model with a piecewise density function that exhibits sharp density contrast and sharp edges (figure 3). The observations are performed on the earth surface in 267 stations aligned on the horizontal direction x_1. We implemented the total variation minimization scheme in a code in MATLAB. The computing time for the inversion is of the order of two minutes in a PC Pentium 300 MHz.

We consider initially only computer noise added during the computing of the numerical solution, avoiding particular noise distributions influencing the inversion results. The regularization parameter was allowed to decrease until no improvement in the solution was obtained due to the presence of low level amplitude noise.

Figure 4 shows an attempt to recover the density function when the norm of the 2D gradient of the solution is used as a stabilization functional. The domain is divided in 240 cells, which turns the problem into highly ill-posed one. The solution is obtained using an adaptive gradient minimization. The starting model was the true model shown in figure 3 with added 25% random noise (starting with the "true" model emphasizes the effect that the regularization term has on the final model). The result of inversion is shown in figure 4. The inverted density function has the correct range of values, but it is very smooth. The inversion procedure fails to recover the non-smooth density distribution in spite of information contained in the starting model. The structural information contained in the starting model was "smoothed" out in the minimization procedure due to the 2D gradient penalty term.

Figure 5 shows the result of inversion with total variation of the density function taken as a stabilization functional. The starting model and discretization used in this simulation were the same as in the previous example. The reconstructed density, though not perfectly blocky, is closer to the non-smooth function. Hence the method is able to recover a contrast density distribution preserving information available in the starting model without "smoothing" it in the stabilized inversion procedure.

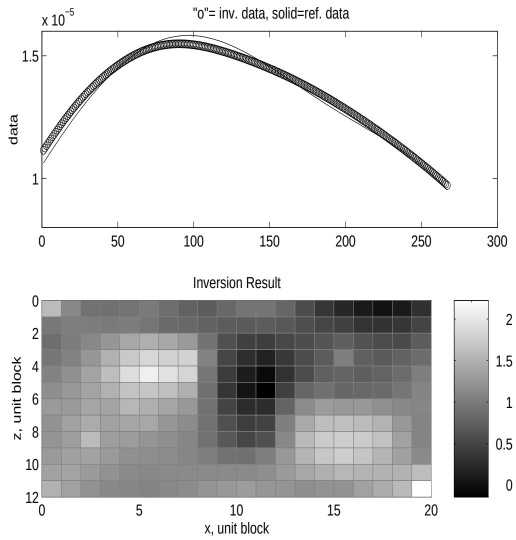

Figure 5: Solution for the 2D total variation scheme.

Conclusion

Stabilization of inversion methods with smoothing functionals can result in oversmoothed reconstructions of Earth's interior properties. We suggest to use the total variation of the model parameters as a stabilization functional to preserve sharp features in inverted models. Examples show that this approach can recover sharp contrast density functions. The adaptive gradient minimization shows to be an efficient scheme for this problem. The approach is especially effective in situations when prior information about the geological structure is available.

Acknowledgement

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References

Bassrei, A., 1993, Regularization and inversion of 2D Gravity Data: 63th. SEG Ann. Mtg., Expand. Abstracts.

Bertete-Aguirre, H., and A. Xavier, 1994, Mínimos quadrados aplicados à inversão geofísica: Coppe-Sistemas Report, UFRJ.

Bertete-Aguirre, H., 1997, Reconstruction of non-smooth solutions of ill-posed geophysical inverse problems, Ph.D. dissertation, COPPE, UFRJ.

Bertete-Aguirre, H., Cherkaeva, E., and Fogelson, A., 1997, Comparison between total variation regularization operators and standard regularization operators for gravimetric inverse problems: Presented at the IV-th SIAM Conf. Math. & Comput. Issues Geosciences.

Chan, T.F., Golub, G.H., and Mulet, P., 1996, A nonlinear primal-dual method for total variation-based image restoration: ICAOS'96, Paris, 241-252.

Constable, S.C., Parker, R.L., and Constable, C.G., 1987, Occam's inversion: A practical algorithm for generating smooth models from electromagnetic sounding data: Geophysics, 52, 289-300.

Dobson, D. and Santosa, F., 1994, An image-enhancement technique for electrical impedance: Inverse Problems, 10, 317-334.

Giusti, E., 1984, Minimal surfaces and functions of bounded variation: Birkhauser.

Lavrentiev, M.M., Romanov, V.G., and Shishatsky, S.P., 1980, Ill-posed problems in mathematical physics and analysis: Nauka, Moscow.

Rudin, L.I., Osher, S., and Fatemi, E., 1992, Nonlinear total variation based noise removal algorithms: Physica D, 60, 259-268.

Telford, W.M., Geldart, L.P., and Sheriff, R.E., 1990, Applied geophysics: Cambridge University Press.

Tikhonov, A.N., and Arsenin, V.Y., 1977, Solutions of ill-posed problems: Willey, New York.

Vogel, C. R., and Oman, M. E., 1996, Iterative methods for total variation denoising: SIAM J. Sci. Comput. 17, 1, 227-238.