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Principal Compliance and Robust Optimal Design

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Abstract. The paper addresses a problem of robust optimal design of elastic structures when the loading is unknown and only an integral constraint for the loading is given. We propose to minimize the principal compliance of the domain equal to the maximum of the stored energy over all admissible loadings. The principal compliance is the maximal compliance under the extreme, worst possible loading. The robust optimal design is formulated as a min–max problem for the energy stored in the structure. The maximum of the energy is chosen over the constrained class of loadings, while the minimum is taken over the design parameters. It is shown that the problem for the extreme loading can be reduced to an elasticity problem with mixed nonlinear boundary conditions; the last problem may have multiple solutions. The optimization with respect to the designed structure takes into account the possible multiplicity of extreme loadings and divides resources (reinforced material) to equally resist all of them. Continuous change of the loading constraint causes bifurcation of the solution of the optimization problem. It is shown that an invariance of the constraints under a symmetry transformation leads to a symmetry of the optimal design. Examples of optimal design are investigated; symmetries and bifurcations of the solutions are revealed.

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This paper is dedicated to the memory of Professor Clifford Truesdell.

1. Introduction

A typical structural optimization problem asks for a material layout in the stiffest design. The stiffness is defined as an elastic energy of a domain loaded by external boundary forces (loading). If the loading is fixed and known, an optimal structure adapts itself to resist the loading. However, the optimal designs are usually unstable to variations of the forces. This instability is a direct result of optimization: To best resist the given loading, all the resistivity of the structure is concentrated against a certain direction thus decreasing its ability to sustain loadings in other directions [7, 8, 20]. For example, consider a problem of optimal design of a structure of a cube of maximal stiffness made from an elastic material and void; assume that the cube is supported on its lower side and loaded by a homogeneous vertical force on its upper side. It is easy to demonstrate, that the optimal structure is a periodic array of unconnected infinitely thin cylindrical rods. Obviously, this design does not resist any other but the vertical loading.

The instability to variations of the loading is not a defect of an optimization procedure – the structure does exactly what it is asked to do; it is a defect of the modeling. In order to find a more stable robust solution, one needs to optimize a more general robust stiffness-like functional that characterizes an elastic body loaded by unspecified (or partly unspecified) forces on its boundary, as it happens with most engineering constructions. To avoid this vulnerability of the optimally designed structures to variations of loading, we propose to minimize *the principal compliance* of the domain equal to the maximum of the stored energy over all admissible loadings. The principal compliance is the maximal compliance under the extreme, worst possible loading. We formulate the robust optimal design problem as a min–max problem for the energy stored in the domain, where the inner maximum is taken over the set of admissible loadings and the minimum is chosen over the design parameters characterizing the structure. This formulation corresponds to physical situations when biological materials are created and engineering constructions are designed to withstand loadings that are not known in advance.

This approach to the structural optimization was discussed in our papers [9, 12] and (for the finite-dimensional model) in the papers [18, 19]. Various aspects of the optimal design against partly unknown loadings were studied in [1, 5, 8, 21, 25–27, 31, 32, 37], see also references therein. In some cases, the minimax design problem, where the designed structure is chosen to minimize maximal compliance of the domain, can be formulated as minimization of the largest eigenvalue of an operator. The minimization of dominant eigenvalues was considered in a setting of the inverse conductivity problem in [11, 13]. The multiplicity of optimal design that we find in the minimax loading-versus-design problem is similar to multiplicity of stationary solutions investigated in the engineering problems of the optimal design against buckling [14, 34] and vibration [30, 28, 33, 22].

The structure of this paper is as follows. In Section 2, we introduce an integral quantity of an elastic domain, *the principal compliance,* equal to the response of the domain to the worst (extremal) boundary loading from the given class of loadings; this quantity is a basic integral characteristic of the domain similar to the capacity, the eigenfrequency, or the volume. The principal compliance is a solution of a variational problem, which can be reduced to an eigenvalue problem or to a bifurcation problem.

Examples of various constraints for admissible loadings and resulting variational problems are considered in Section 3. Particularly, the variational problem for the principal compliance with a quadratically constrained class of loadings is reduced to the Steklov eigenvalue problem. The principal compliance of the domain in this case is a reciprocal of the principal Steklov eigenvalue. We also consider the constraints of the L_p norm, $p > 1$, of the loading and inhomogeneous constraints and show that the L_p norm constraints result in a nonlinear boundary value problem. The constraint of L_1 norm of the loading yields to a variational problem which does not have a classical solution, but a distribution: the optimal loading turns out to be a δ-function or, physically speaking, a concentrated loading (if such a loading does not lead to infinite energy).

Section 4 considers robust structural optimization which is formulated as a problem of minimization of the principal compliance. The optimal design takes into account the multiplicity of stationary solutions for extreme (most dangerous) loadings; typically, the optimal structure equally resists several extreme loadings. The set of the extreme loadings depends on the constraints of the problem. Continuous change of the constraints leads to modification of the set of extreme loadings; the optimal structure changes in response. This corresponds to bifurcation of the solution of the optimization problem. Another characteristic feature of the optimization problem is the symmetry of its solution. We show that the invariance of the set of the constraints for the admissible loadings, together with the corresponding symmetry of the domain, leads to the symmetry of the optimally designed structure.

Section 5 contains two examples of problems of structural design for uncertain loadings. One example is provided by the problem of designing the optimally supported beam loaded by an unknown loading with fixed mean value. The second example is a problem of determining the optimal structure of a composite strip loaded by a force which deviates from the normal in an unknown direction. The force is assumed to have a prescribed normal component and an additional component which is arbitrarily directed and is unknown.

2. The Principal Compliance of a Domain

2.1. PROBLEM, EQUATIONS, CONSTRAINTS

2.1.1. *Equations*

Consider a domain Ω with the boundary $\partial\Omega = \partial_0 \cup \partial$ filled with a linear anisotropic elastic material, loaded on its boundary component ∂ by a force f, and fixed on the boundary component ∂_0. The elastic equilibrium of such a body is described by a system (see, for instance, [35]):

$$\begin{aligned} &\nabla \cdot \sigma = 0 \quad \text{in } \Omega, \qquad \sigma = C : \epsilon, \\ &\sigma = \sigma^{\mathrm{T}}, \qquad \epsilon(w) = \frac{1}{2}\big(\nabla w + (\nabla w)^{\mathrm{T}}\big). \end{aligned} \tag{1}$$

Here $C = C(x)$ is the fourth-order stiffness tensor of an anisotropic inhomogeneous material, $w = w(x)$ is the displacement vector, ϵ is the strain tensor, σ is the stress tensor, and (:) represents contraction of two indices. Thus,

$$\epsilon : \sigma = \sum_{i,j} \epsilon_{ij}\sigma_{ji}, \qquad (C : \epsilon)_{ij} = \sum_{k,l} C_{ijkl}\epsilon_{lk}.$$

Equation (1) is supplemented with the boundary conditions

$$\sigma \cdot n = f \quad \text{on } \partial, \qquad w = 0 \quad \text{on } \partial_0, \tag{2}$$

where n is the normal to the boundary $\partial\Omega$. These equations are the first variation conditions of the variational problem,

$$\begin{aligned} \mathcal{J}(C, f) &= -\min_{w: w|_{\partial_0}=0} \left(\int_\Omega \Pi(C, \epsilon(w))\, \mathrm{d}x - \int_\partial w \cdot f \, \mathrm{d}s \right) \\ &= \max_{w: w|_{\partial_0}=0} \left(\int_\partial w \cdot f \, \mathrm{d}s - \int_\Omega \Pi(C, \epsilon(w))\, \mathrm{d}x \right), \end{aligned} \tag{3}$$

where Π is the density of the elastic energy:

$$\Pi(C, \epsilon(w)) = \frac{1}{2}\epsilon : \sigma = \frac{1}{2}\epsilon : C : \epsilon. \tag{4}$$

The nonnegative functional $\mathcal{J}$ is called the *compliance* of the domain; (3) states that it is maximal at equilibrium. At equilibrium, the energy stored in the body equals the work of the applied external forces f,

$$\mathcal{J}_0(C, f) = \frac{1}{2} \int_\partial w \cdot f \, \mathrm{d}s = \int_\Omega \Pi(C, \epsilon(w))\, \mathrm{d}x. \tag{5}$$

Simultaneously with the elasticity problem, we consider also a close problem of the bending of a Kirchhoff plate (see, for example, [35]). The equilibrium of the plate is described by the fourth order equation

$$\nabla\nabla : C_{pl} : \nabla\nabla w = f \quad \text{in } \Omega \tag{6}$$

with homogeneous boundary conditions

$$w = 0 \quad \text{on } \partial\Omega, \qquad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{7}$$

corresponding to a clamped plate, or

$$w = 0 \quad \text{on } \partial\Omega, \qquad n^{\mathrm{T}}(C_{pl} : \nabla\nabla w)n = 0 \quad \text{on } \partial\Omega, \tag{8}$$

for a simply supported plate. Here, w is the deflection orthogonal to the plane of the plate, C_{pl} is the fourth-order tensor of bending stiffness of the elastic material, $\nabla\nabla w$ is the Hessian of w, and f is the external loading. Notice that the force f enters the equation as a right-hand-side term. The equation for the plate deflection corresponds to maximization of the functional

$$\mathcal{J}_{pl}(C, f) = -\int_\Omega \left(\frac{1}{2} \nabla\nabla w : C_{pl} : \nabla\nabla w - wf \right) \mathrm{d}x. \tag{9}$$

The results that we develop further in this paper apply to both the elasticity (1) and the bending problem (6); therefore, we will drop the subscript in $\mathcal{J}_{pl}(C, f)$, and keep notation $\mathcal{J}(C, f)$ for both compliance functionals. If this does not cause a confusion, we use the same notation w to denote both the displacement in the elasticity problem (1) and the deflection in the bending problem (6), even though the first one is a vector function, whereas the second one is a scalar function.

2.1.2. *Admissible Loadings*

Let $\mathcal{F}$ be a set of admissible loadings f. The elastic energy over a finite domain is assumed to be finite. We consider integral constraints to describe the set of loadings $\mathcal{F}$:

$$\mathcal{F} = \left\{ f\colon \int_{D_f} \phi(f)\,\mathrm{d}s = 1 \right\}, \qquad D_f = \begin{cases} \partial, & \text{for problem (1)}, \\ \Omega, & \text{for problem (6)}. \end{cases} \tag{10}$$

Here D_f is a domain of application of the forces: in the elasticity problem (1), D_f concides with the part of the boundary ∂, whereas for the bending plate problem (6), D_f is the domain Ω or a part of it. We assume that ϕ is a convex function of f, with the derivative $\psi\colon \mathbb{R}^3 \to \mathbb{R}^3$:

$$\psi(f) = \frac{\partial \phi}{\partial f} = \left(\frac{\partial \phi}{\partial f_1}, \frac{\partial \phi}{\partial f_2}, \frac{\partial \phi}{\partial f_3} \right),$$

which has an inverse $\rho = \psi^{-1}$.

2.1.3. *Principal Compliance*

We define the *principal compliance* of an elastic domain in a class of loadings as a compliance in the worst possible loading scenario.

DEFINITION. The principal compliance Λ of the domain is

$$\Lambda = \max_{f \in \mathcal{F}} \mathcal{J}(C, f). \tag{11}$$

The loadings that correspond to the principal compliance Λ are *extreme* or *the most dangerous loadings*; we denote them as f_D.

$$\Lambda(C) = \mathcal{J}(C, f_D) \geqslant \mathcal{J}(C, f) \quad \forall f \in \mathcal{F}. \tag{12}$$

The most dangerous loadings exist if the set $\mathcal{F}$ is closed and convex, see [15].

2.2. CALCULATION OF THE PRINCIPAL COMPLIANCE

The concept of the principal compliance is useful if there are efficient algorithms for computing the extreme loadings. We show here that the problem of computation of the principal compliance and the extreme loadings can be formulated as a boundary value problem.

Consider problem (11) and assume that the loadings are constrained as in (10). The augmented functional J for the problem is:

$$J = \mathcal{J}(C, f) - \mu\left(\int_{D_f} \phi(f)\,\mathrm{d}s - 1\right),$$

where μ is the Lagrange multiplier. Clearly, $\max_{f\in\mathcal{F}} \mathcal{J} = \max_f J$. Variation of the augmented functional with respect to f gives the optimality condition for the extreme loading(s):

$$\delta_f J = \int_{D_f} \frac{\partial}{\partial f}(-f\cdot w + \mu\phi(f))\delta f = 0,$$

or, since δf is arbitrary,

$$w - \mu\frac{\partial\phi}{\partial f} = 0 \quad \text{on } D_f.$$

Solving for the extreme loading(s) $f_D = f$, we arrive at the condition

$$f_D = \rho\left(\frac{w}{\mu}\right) \tag{13}$$

which links the loading f_D to the displacement w at the same boundary point for the elasticity problem (1) or at the same point in the domain for the bending problem. Condition (13) together with the first boundary condition in (2) allows us to exclude f from the boundary conditions, leading to the boundary value problem for the displacement w. We arrive at:

THEOREM 1. *The principal compliance* Λ *of the elasticity problem* (1), (2) *with the constraints for the class of loadings* (10) *equals*

$$\Lambda = \frac{1}{2}\int_{\partial} w\rho\left(\frac{w}{\mu}\right)\mathrm{d}s, \tag{14}$$

where w *satisfies the elasticity equations* (1) *in* Ω *with the boundary conditions*

$$\sigma\cdot n = \rho\left(\frac{1}{\mu}w\right) \quad \text{on } \partial, \qquad w = 0 \quad \text{on } \partial_0. \tag{15}$$

The Lagrange multiplier μ *is determined from the integral condition*

$$\int_{\partial}\phi\left(\rho\left(\frac{w}{\mu}\right)\right)\mathrm{d}s = 1, \tag{16}$$

where the function $\rho(\cdot)$ *is an inverse of* $\psi = \partial\phi/\partial f$.

Indeed, the displacement w, whose energy is the principal compliance, satisfies the elasticity equations (1) in Ω with the boundary conditions obtained from (2) and (13). The first condition in (15) relates the normal stress at a point on the boundary ∂ to the displacement at this point. The boundary value problem (1), (15), (16) allows us to compute w and μ, f_D, and Λ.

For the bending problem (6), the calculation is similar. The principal compliance is the maximum of the functional (9) over all loadings bounded by the constraint (10); its value is the following.

THEOREM 2. *The principal compliance* Λ *for the bending problem* (6)–(8) *with the constraint for the class of loadings* (10) *is*

$$\Lambda = \frac{1}{2}\int_{\Omega} w\rho\left(\frac{w}{\mu}\right) \mathrm{d}x, \tag{17}$$

where w *satisfies the equation*

$$\nabla\nabla : C_{pl} : \nabla\nabla w = \rho\left(\frac{w}{\mu}\right) \tag{18}$$

together with the corresponding homogeneous boundary conditions (7) *or* (8). *The function* $\rho(\cdot)$ *is an inverse of* $\psi = \partial\phi/\partial f$. *The Lagrange multiplier* μ *is determined from*

$$\int_{\Omega} \phi\left(\rho\left(\frac{w}{\mu}\right)\right) \mathrm{d}s = 1. \tag{19}$$

Indeed, the extreme loading f is related to the displacement w by a scalar relation $w = \mu\phi'(f)$ or $f = \rho(w/\mu)$, and the plate equlibrium is described by equation (18).

3. Examples of Constraints

3.1. HOMOGENEOUS QUADRATIC CONSTRAINT

Assume that the constraint (10) restricts a weighted L_2 norm of f:

$$\frac{1}{2}\int_{\partial} f^{\mathrm{T}}\Psi f\, \mathrm{d}s = 1 \quad \text{or} \quad \phi(f) = \frac{1}{2} f^{\mathrm{T}}\Psi f, \tag{20}$$

where $\Psi(s)$ is a symmetric, positive matrix. In this case, ρ is a linear mapping: $\rho(f) = \Psi^{-1} f$, and the first of the boundary conditions (15) for the extremal loading becomes linear:

$$\frac{1}{\mu}\Psi^{-1} w - \sigma \cdot n = 0 \quad \text{on } \partial. \tag{21}$$

The optimality condition states that w and $\sigma \cdot n$ are proportional to each other everywhere on the boundary ∂ with the same tensor of proportionality $\mu\Psi$.

REMARK 1. The stationary condition (21) allows for the following physical interpretation: The boundary ∂ is equipped with distributed springs with negative stiffness. The forces in them are proportional but opposite to the forces in conventional linear springs.

The elasticity equations (1) with boundary conditions (21) form a linear *eigenvalue problem* that has a nonzero solution w only if $1/\mu$ is one of its discrete eigenvalues. Eigenvalue $1/\mu$ relates the displacement on the boundary and the normal stress.

As all eigenvalue problems, the problem (1), (21) represent Euler–Lagrange equations of a variational problem:

$$\frac{1}{\mu} = \min_{w: w|_\partial = 0} \frac{\int_\Omega \epsilon(w) : C : \epsilon(w)\,\mathrm{d}s}{\int_\partial w \cdot \Psi^{-1} w\,\mathrm{d}s}$$

or

$$\left(\int_\Omega \epsilon(w) : C : \epsilon(w)\,\mathrm{d}x - \frac{1}{\mu} \int_\partial w \cdot \Psi^{-1} w\,\mathrm{d}s \right) \to \min_{w: w|_\partial = 0}. \tag{22}$$

The eigenvalue problem that contains the eigenvalue in the boundary condition is a *Steklov eigenvalue problem*, and μ is a reciprocal to the Steklov eigenvalue, see [4]. The eigenfunctions are normalized by condition (20).

Using (20) and (21) in the form $w = \mu \Psi f$, we observe that the second term in (22) is equal to μ, thereafter $\mu = \Lambda$. The Steklov problem has infinitely many real positive eigenvalues (see [4, 23]), but the principal compliance of the domain corresponds to the dominant eigenvalue, $\Lambda = \mu_{\max}$. The dominant eigenfunction is not necessarily unique; we will demonstrate below that the existence of many stationary solutions is typical for the problems of minimization of the principal compliance with respect to the structure. The dominant eigenfunctions are the extreme loadings. The results are formulated as

THEOREM 3. *If the L_2-norm of admissible loadings is bounded, the principal compliance Λ is a solution of the eigenvalue problem:*

$$\nabla \cdot \sigma = 0 \quad in\ \Omega, \qquad w = \Lambda \Psi \sigma \cdot n \quad on\ \partial. \tag{23}$$

Λ is a reciprocal to the principal eigenvalue $1/\mu$ of the problem (1), (21).

REMARK 2. The spectrum of the problem (1), (21) has one condensation point, zero. Positive eigenvalues μ_k tend to zero but never reach it. This implies that the dual problem of *minimal* compliance does not have a solution: the compliance can be made arbitrarily small by choosing a fast alternating loading.

REMARK 3. The problem becomes isomorphic to the problem of the principal eigenfrequency of the domain, if the kinetic energy (and the inertia) are concentrated on the boundary: $T = \delta(x - x_b) w \Psi w$, where $x_b \in \partial$.

In the bending problem (6), the analogy between the principal compliance and the principal eigenfrequency of vibrations is complete. The equilibrium (18) of the optimally loaded plate coincides with the equation for the magnitude of the deflection of the oscillating plate,

$$\nabla\nabla : C_{pl} : \nabla\nabla w = \frac{1}{\Lambda} w.$$

3.2. L_1-NORM CONSTRAINT

Consider the L_1-norm constraint for the class of admissible loadings which assumes that the mean value of loading's magnitude is fixed:

$$\int_\partial |f| \, \mathrm{d}s = \int_\partial \sqrt{f \cdot f} \, \mathrm{d}s = 1. \tag{24}$$

From an engineering viewpoint, this case is probably the most interesting one: it models the situation when the total weight applied to the structure is known but the distribution of the loading over the boundary is uncertain.

For this, the functional of the variational problem grows linearly as $|f| \to \infty$ which leads to a significantly different analysis. The straightforward variational technique does not provide the correct answer. Indeed, the variation with respect to f returns the vector condition

$$\delta f: \quad w - \mu \frac{1}{\sqrt{f \cdot f}} f = 0 \quad \text{on } \partial,$$

which says that

$$|w| = \text{constant} \quad \text{and} \quad w \parallel f \quad \text{on } \partial.$$

The last condition, together with the condition $\sigma \cdot n = f$ (see (2)), allows us to exclude f and end up with a pair of conditions on w:

$$(\sigma \cdot n) \times w = 0, \quad |w| = \text{constant} \quad \text{on } \partial.$$

Generally, these conditions cannot be satisfied if the ∂-component of the boundary is adjacent to the component ∂_0 where $w = 0$ since w is continuous. This contradiction shows that the naive variational method does not apply.

REMARK 4. The appearance of discontinuous solutions in the variational problems of linear growth is well-known [36]. The famous classical example is the existence of a non-smooth solution in the minimal surface problem.

To solve the contradiction, we need to assume that the optimal loading f is a distribution. Indeed, the distribution does not have to satisfy the Euler equations of the variational problem because this equation was derived under the assumption that the optimal solution f is finite and smooth.

Dealing with distributions in the L_1-constrained set of loadings may cause difficulties because the distributions $\delta(x - x_0)$ may or may not correspond to a finite energy of the elastic system, as is stated in the Sobolev embedding theorem, see, for example, [24]. For the compliance of the bending plate (9), the energy of the concentrated loading and the Green's function of the corresponding operator are finite. We illustrate this case below considering a one-dimensional example of a beam; the concentrated loadings of the type $\delta(x - x_0)$ are acceptable because the corresponding energy stored in the elastic beam is finite.

However, the linear elasticity problem does not allow a concentrated loading because the corresponding energy is infinite; the Green's function $g(x, y)$ has a singularity, $g(x, x) = \infty$. In this case, the restriction on the class of admissible f can be slightly tighten. We may assume, for example, that the force is piecewise constant within small domains of area ϵ. Alternatively, we may constrain the $L_{1+\epsilon}$-norm of the loading,

$$\int_\partial |f|^{1+\epsilon}\,\mathrm{d}s = 1, \tag{25}$$

where $\epsilon > 0$ is a fixed parameter. This loading can be supported by a linear elastic material, although the displacement w can indefinitely grow when $\epsilon \to 0$. The analysis of this case leads to the optimality condition

$$f = \left|\frac{w}{\mu}\right|^{1/\epsilon} \frac{w}{|w|},$$

which shows that magnitude of an optimal loading either stays arbitrarily close to zero or is very large (of the order of $1/\epsilon$). The integral constraint (25) guarantees that the measure of the set of large values of $f(s)$ goes to zero when $\epsilon \to 0$.

With this warning, we proceed with the formal analysis of the problem with the L_1 constraint assuming that either the limit exists or that ϵ can be chosen arbitrary close to zero to preserve the qualitative properties of the solution.

The extremal loading is concentrated in several points,

$$f = \sum_i c_i \xi_i \delta(x - x_i),$$

where $\{x_i\}$ is the set of points where the (concentrated) loading is applied, $x_i \in \partial$, $\xi_i : \xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, \xi_i^{(3)})$, $|\xi_i| = 1$, are directional vectors of the concentrated loadings, and c_i are their intensities; due to (24), c_i belong to the simplex

$$c_i\colon \sum_i c_i = 1, \quad c_i \geqslant 0. \tag{26}$$

Further, we show that the extreme loading is always applied to a single point. The displacements $w_k = w(x_k)$ are

$$w_k = \sum_i g(x_k, x_i) c_i \xi_i,$$

where $g(x_k, x_i)$ is the Green's function which relates the δ-function loading at the point x_i to the generated displacement w at the point x_k. The compliance becomes

$$\mathcal{J} = \sum_i \sum_k c_i c_k \left(\xi_i^{\mathrm{T}} g(x_i, x_k) \xi_k\right).$$

The principal compliance corresponds to the maximum of $\mathcal{J}$ with respect to c_i, ξ_i and the points x_i.

As a function of c_i, $\mathcal{J}$ is a nonnegative quadratic form, because the work $\mathcal{J}$ is always nonnegative. Therefore, $\mathcal{J}$ is a convex function of c_i and its maximum is reached in a corner of the simplex (26): the maximum $\mathcal{J}_c$ of $\mathcal{J}$ corresponds to a single concentrated loading $c_1 = 1, c_2 = \cdots = c_p = 0$. Next, we maximize this maximum $\mathcal{J}_c$ with respect to the direction $\xi_1 = (\xi_1^{(1)}, \xi_1^{(2)}, \xi_1^{(3)})$ of the single applied loading. The resulting compliance $\mathcal{J}_{\xi,c}$ is equal to the maximal eigenvalue $\lambda_{\max}^g(x_1)$ of the Green's function $g(x_1, x_1)$ at the point $x = x_1$:

$$\mathcal{J}_{\xi,c} = \max_{\xi_1}\left(\xi_1^{\mathrm{T}} g(x_1, x_1)\xi_1\right) = \lambda_{\max}^g(x_1).$$

This implies that the applied loading $f(x)$ must be parallel to the displacement $w(x)$. Finally, we choose the point $x_1 \in \partial$ of application of the extreme concentrated loading and obtain the principal compliance Λ. Summarizing, we obtain

THEOREM 4. *The L_1-principal compliance is*

$$\Lambda = \max_{x \in \partial}\left\{\lambda_{\max}^g(x)\right\},$$

where $\lambda_{\max}^g(x)$ is the maximal eigenvalue of the 3×3 *tensor Green's function $g(x, x)$ of the problem* (1) *at the point $x \in \partial$.*

We stress that the point x_1 may be not unique although the extreme loading is always concentrated at one point. For example, there may be two symmetric extreme loadings if Ω is a symmetric domain. An example in Section 5.1 below shows that there are several equally dangerous loadings in an optimal solution: $\lambda_{\max}^g(x_1) = \cdots = \lambda_{\max}^g(x_q)$; the number q depends on the structure.

3.3. OTHER SPECIAL CASES

3.3.1. *Constrained L_p-norm of the Loading*

If the constraint is imposed on the L_p-norm of the loading, i.e.,

$$\frac{1}{p}\int_\partial |f|^p = 1, \quad p > 1,$$

the problem has the form (1) but the boundary conditions (21) are replaced by

$$\sigma \cdot n = \eta(w), \quad \eta(w) = \left(\frac{|w|}{\mu}\right)^{1/(p-1)} \frac{w}{|w|} \tag{27}$$

and the normalization (16) for μ becomes

$$\mu = \left(\frac{1}{p}\int_{\partial}|w|^q\,\mathrm{d}s\right)^{1/q} \quad \text{with } \frac{1}{q}+\frac{1}{p}=1. \tag{28}$$

In this case, the relation between the stress and displacement is nonlinear. Again, the multiplicity of stationary solutions that satisfy (27), (28) is expected; this time the solutions correspond to bifurcation points instead of spectrum points. The physical interpretation is similar to the one given in Remark 1, but the springs attached to the boundary ∂ are nonlinear.

3.3.2. *Nonhomogeneous Constraint*

Let the loading f consist of some known component f^0 and an unknown deviation with a constrained L_p-norm:

$$\|f^0 - f\|_{L_p} \leqslant 1. \tag{29}$$

Applying the previous variational analysis, we conclude that an extremal loading can be found from the elasticity problem with a inhomogeneous mixed boundary condition:

$$\sigma \cdot n = f^0 + \eta(w) \quad \text{on } \partial.$$

Since the boundary condition is inhomogeneous, $w = 0$ is not a solution. Still, the problem may have several stationary solutions. An example of this constraint is discussed later in Section 5.2.

4. Robust Optimal Design

4.1. MULTIPLICITY OF EXTREME LOADINGS

Consider an optimal design problem: find a layout of elastic materials over the domain Ω that minimizes the principal compliance Λ. Such a structure (stiffness $C(x)$) corresponds to a solution of the extremal problem

$$P_{\min\max} = \min_{C\in\mathcal{C}} \Lambda(C), \tag{30}$$

where $\mathcal{C}$ is a class of admissible layouts. We rewrite the problem using the definition of $\Lambda(C)$:

$$P_{\min\max} = \min_{C\in\mathcal{C}} \max_{f\in\mathcal{F}} \mathcal{J}(C, f), \tag{31}$$

where the compliance $\mathcal{J} = \mathcal{J}(C, f)$ is defined in (3). Minimization over w in (3) is performed first so that w will satisfy the elasticity equations while interchanging the order of the extremal operations $\min_{C\in\mathcal{C}}$ and $\max_{f\in\mathcal{F}}$ correspond to two physically different situations. Minimax problem (31) is a problem of optimization of the material layout when the applied loading is unknown, while in the maximin problem

$$P_{\max\min} = \max_{f\in\mathcal{F}} \min_{C\in\mathcal{C}} \mathcal{J}(C, f) \tag{32}$$

the loading is chosen to maximize the stored energy and is known to the designer; so the design resists this particular loading. If $\mathcal{J}$ is a saddle-point functional, the solutions to these two problems coincide, and

$$P_{\max\min} = P_{\min\max}.$$

Saddle point solutions are typical for 'weak' control as we will demonstrate below. The general case

$$P_{\max\min} < P_{\min\max}$$

corresponds to a situation when several loadings are 'equally dangerous.' The stiffness of the structure C_{opt} should be fairly distributed to resist equally well each of these extreme loadings leading to the condition

$$\mathcal{J}(C_{\text{opt}}, f_i) = \mathcal{J}(C_{\text{opt}}, f_j), \quad f_i, f_j \in \Phi,$$

where Φ is a set of extreme loadings.

Generally, the set of stationary loadings may consist of any number of elements. They can be found from the following equations, see [16]. Consider a design C_{opt} and the functional $\mathcal{J}(C_{\text{opt}}, f)$. The extremal loadings that solve the variational problem

$$\frac{\delta}{\delta f}\mathcal{J}(C_{\text{opt}}, f) = 0, \qquad \frac{\delta^2}{\delta f^2}\mathcal{J}(C_{\text{opt}}, f) \leqslant 0$$

are denoted by $\hat{f}_i$, $i = 1, \dots, p$, where $p \leqslant \infty$; we assume that there are p stationary loadings that can become extreme. The optimized principal compliance $P_{\min\max}$ is determined from the problem

$$\min_{C} \max_{\nu_i\geqslant 0}\left(P_{\min\max} + \sum_i \nu_i \mathcal{J}(C_{\text{opt}}, \hat{f}_i)\right), \tag{33}$$

where $\nu_i \geqslant 0$ are the Lagrange multipliers due to the constraints

$$\mathcal{J}(C_{\text{opt}}, \hat{f}_i) - P_{\min\max} \leqslant 0, \qquad \sum_i \nu_i = 1.$$

Optimal design C_{opt} is found from the following conditions that reformulate the minimax problem as the problem of minimization of a sum of energies corresponding to extreme loadings.

THEOREM 5. *The optimal principal compliance $P_{\min\max}$ equals*

$$P_{\min\max} = \min_{C \in \mathcal{C}} \max_{\{\nu_i\}:\nu_i>0} \sum_{i=1}^{q} \nu_i \mathcal{J}(C, \hat{f}_i), \qquad \sum_i \nu_i = 1, \tag{34}$$

where q is the number of active extreme loadings.

The nonzero Lagrange multipliers correspond to the equalities

$$\mathcal{J}_0 = \mathcal{J}(C_{\text{opt}}, \hat{f}_i) \quad i = 1, \ldots, q, \quad \Rightarrow \quad \nu_i > 0,$$

and the multipliers equal zero if the stationary loading leads to a smaller value of the functional, i.e.,

$$\mathcal{J}_0 > \mathcal{J}(C_{\text{opt}}, \hat{f}_k), \quad k = q+1, \ldots, p \quad \Rightarrow \quad \nu_k = 0.$$

These last conditions should be checked in the optimization procedure; that is, minimizing $\mathcal{J}_0$ we check if the value of the functional for the next loading f_{q+1} (not the most dangerous one) is still less than $\mathcal{J}_0$. When this inequality becomes equality, the set of extreme loadings should be enlarged to include $\hat{f}_{q+1}$, and the corresponding Lagrange multiplier ν_{q+1} becomes positive.

The multiplicity of equally dangerous loadings closely resembles the multiplicity of optimal solutions in a well studied problem of maximization of the minimal eigenfrequency. The multiplicity of optimal eigenvalues in that problem was observed first in a pioneering paper of Olhoff and Rasmussen [30]; then it was investigated in [33, 14, 34].

REMARK 5. The optimization problem (34) also admits a probabilistic interpretation. Namely, assume that the optimal loading is a random variable which takes q stationary values with some probability $\nu_1, \ldots, \nu_q$. Then the sum $\sum \nu_i \mathcal{J}(C, f_i)$ in (34) is the expectation of the energy. The optimal design minimizes the expectation of the energy, meanwhile the loading chooses probabilities $\nu_1, \ldots, \nu_q$ to maximize it.

4.2. SYMMETRIES

Symmetries are typical for designs that minimize the principal compliance. Namely, if the domain and the class of loadings are invariant under a symmetry transformation (translation, reflection, or rotation), then the set of extreme loadings Φ and the optimal design are invariant under this transformation as well. We state the following:

THEOREM 6. *If the domain Ω, the boundary component ∂, and the set $\mathcal{F}$ of admissible loadings are invariant under a symmetry transformation $\mathcal{R}$, i.e.,*

$$\Omega = \mathcal{R}\Omega, \qquad \partial = \mathcal{R}\partial, \quad \textit{and} \quad \mathcal{F} = \mathcal{R}\mathcal{F},$$

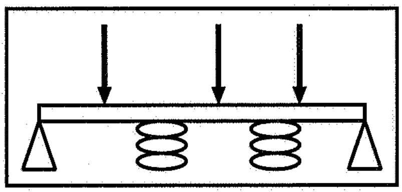


Figure 1. The force could be applied at arbitrary points along the elastically supported beam. The mean value of the magnitude of the force is constrained.

then the set of extreme loadings Φ and the optimal materials' layout C are invariant under this transformation, i.e.,

$$\Phi = \mathcal{R}\Phi, \qquad C = \mathcal{R}C. \tag{35}$$

Indeed, applying the above consideration we can see that if $f_0 \in \Phi$ is an extreme loading, then $\mathcal{R} f_0$ is also an extreme loading. The compliance of the structure should be the same for both loadings, which implies invariance of the design parameters with respect to the transformation $\mathcal{R}$. Particularly, when the loaded domain is rotationally symmetric, and the loading can be applied from any direction, the optimal layout is axisymmetric.

REMARK 6. Notice the symmetry of many natural 'designs' that are perfected by evolution: The rotationally symmetric shape of trees allows them to sustain wind from all directions; our natural "protective shell", the skull, provides the best protection for the brain against hits from any direction.

The conditions of the theorem do not require the symmetry of the extreme loading, only a possibility to apply a loading symmetric to any given one. In contrast, the design must be symmetric.

5. Examples of Optimal Designs

The following examples highlight the discussed multiplicity of extreme loadings and bifurcation of the optimal solution.

5.1. OPTIMAL DESIGN OF A SUPPORTED BEAM

5.1.1. *Formulation*

Consider a homogeneous elastic beam of unit length simply supported at both ends, elastically supported from below by a distributed system of elastic vertical springs with the specific stiffness $q(x) \geqslant 0$, and loaded by a distributed nonnegative force $f(x) \geqslant 0$. The elastic equilibrium of the displacement w is described by a one-dimensional version of (6):

$$(Ew'')'' + qw = f, \qquad w(0) = w(1) = 0, \qquad w''(0) = w''(1) = 0, \tag{36}$$

where E is Young's modulus. The compliance is equal to

$$\mathcal{J} = \int_0^1 \left(fw - \frac{E}{2}(w'')^2 - \frac{q}{2}w^2 \right) \mathrm{d}x, \tag{37}$$

where w is a solution of (36). Assume that the mean value of the magnitude of the loading (L_1-norm constraint) is equal to one, and the integral stiffness of the supporting springs is constrained by a constant κ.

$$\begin{aligned} \mathcal{F} &= \left\{ f \in H^{-1}(0,1) \colon \int_0^1 f \, \mathrm{d}x = 1 \right\}, \\ \mathcal{Q} &= \left\{ q \in H^{-1}(0,1) \colon \int_0^1 q \, \mathrm{d}x = \kappa \right\}. \end{aligned}$$

The optimal design problem of minimization of the principal compliance by distributing the springs stiffness becomes:

$$P_{\min\max} = \min_{q \in \mathcal{Q}} \left(\max_{f : \in \mathcal{F}} \mathcal{J} \right).$$

Applying the above analysis, we conclude:

1. The domain, class of loadings and the boundary conditions are invariant to the translation $x \to 1 - x$, therefore the design (the springs stiffness) is symmetric with respect to the center of the beam, see Section 4.2,

$$q(x) = q(1-x).$$

2. Necessary conditions in Section 3.2 show that the extreme loading is a delta-function $f(x) = \delta(x - x_j)$ applied at one of the points $\{x_1, x_2, \ldots, x_p\}$, where

$$w'(x_j) = 0, \qquad w''(x_j) \leqslant 0. \tag{38}$$

   The extreme loading may be applied to different points symmetric with respect to the center of the beam; the resulting stiffness must be equal.

3. The stiffness of an optimal spring is a distribution

$$q(x) = \sum_i \alpha_i \delta(x - y_i), \qquad \sum_i \alpha_i = \kappa, \quad \alpha_i \geqslant 0.$$

   Indeed, the assumption that $q(x)$ satisfies variational stationary conditions leads to a contradiction similar to the contradiction discussed in Section 3.2. Particularly, the optimal positions of the springs satisfy the necessary conditions (38), and therefore the set of reinforcement points coincides with the set $\{x_1, x_2, \ldots, x_p\}$. The number p of the critical points depends on the relative stiffness of the springs κ/E.

Accounting for the loading and springs being concentrated, we reformulate the problem (37) for the optimal principal compliance:

$$P_{\min\max} = \min_{(\alpha_1,\dots,\alpha_p)} \max_{x_k} \left\{ \sum_{i=1}^{p} \left(\delta_{ik} w_k - \frac{\alpha_i}{2} w_i^2 \right) - \int_0^1 \frac{E}{2} (w'')^2 \, dx \right\}, \tag{39}$$

where δ_{ik} is Dirac function.

The response of a supported beam can be characterized by a function

$$v(x) = \max_{\zeta \in (0,1)} g(\zeta, x), \tag{40}$$

where g is the Green's function of the boundary value problem (36): $g(\zeta, x)$ is the displacement $w(\zeta)$ at the point ζ corresponding to a delta-function loading applied at the point x, $f(\zeta) = \delta(\zeta - x)$, and $v(x)$ is the maximal displacement under the concentrated force applied at the point x. Figure 2 shows the response $v(x)$ of the beam supported by two symmetric springs. The family of the thin curves shows the displacements $w_k(x)$ under several concentrated loadings applied at different points along the beam. The thick curve shows the maximal displacement, $v(x)$. Notice that the point of application of the concentrated force is generally different from the point of maximum of the displacement curve; see the caption to Figure 2. However, the *optimal* springs are located at points x^i_{opt}, $i = 1, 2$, of the maximum of $v(x)$, and the extreme loading is the one applied at one of the same points, $f_D = \delta(x - x^1_{\text{opt}})$ or $f_D = \delta(x - x^2_{\text{opt}})$.

The numerical results demonstrate the following: if the springs are weak, $\kappa/E \leqslant \kappa_1$, they are concentrated in the center of the beam. We are dealing with the saddle-point case: the most dangerous loading is a concentrated loading applied also at the center. The maximal displacement $v(x)$ is a unimodal function of the position of the loading, with the maximum in the center, $(v'(1/2) = 0, v''(1/2) < 0)$. There

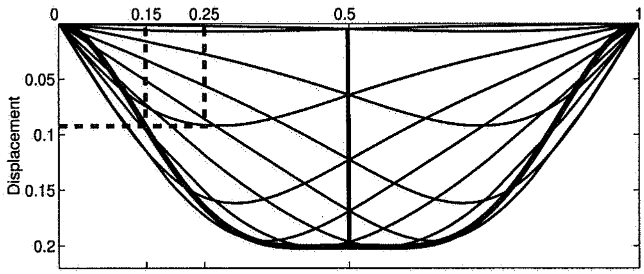
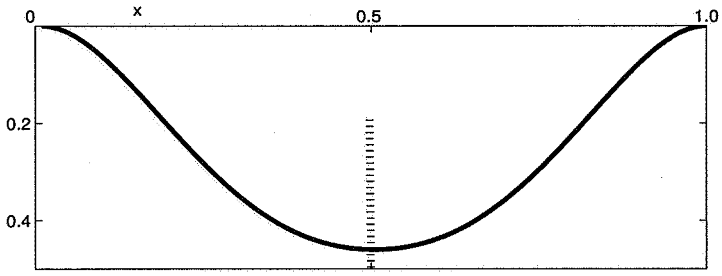
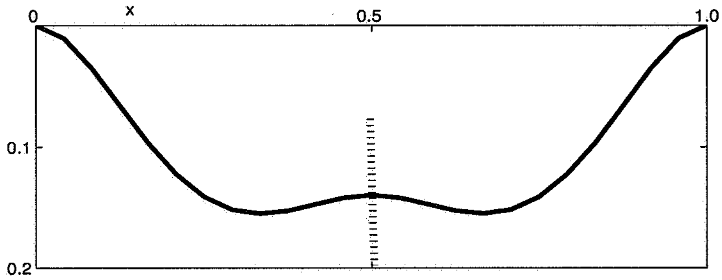


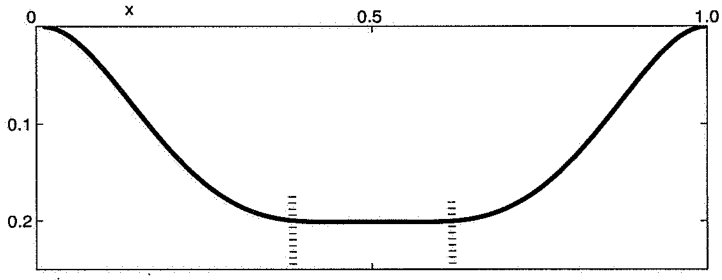
Figure 2. Thin curves: The displacement functions generated by concentrated loadings applied at various points along the beam. The thick curve: maximal displacement $v(x)$ generated by a force applied at $x \in (0, 1)$ as a function of the position of the force. The displacement corresponding to the force applied at $x = 0.15$, has a maximum at $x = 0.25$. Figure shows the responses of the beam optimally reinforced by two symmetric springs.



(a)



(b)



(c)

Figure 3. Maximal displacement $v(x)$ as a function of the position of the applied loading: (a) corresponds to a saddle point case, $\kappa/E < \kappa_1$: the function $v(x)$ is unimodal, the optimal spring and the extreme loading are both located in the middle of the beam; (b) shows $v(x)$ corresponding to κ/E in the interval $\kappa_1 < \kappa/E < \kappa_2$ when the strong spring is located in the center of the beam. Maximal displacement $v(x)$ is not unimodal; design is not optimal; (c) corresponds to κ/E in the same interval $\kappa_1 < \kappa/E < \kappa_2$, the maximal displacement $v(x)$ is shown for an optimally designed beam which is supported by two symmetric springs.

is only one solution for the optimal applied force and the optimal position of the spring:

$$f(x) = \delta\left(x - \frac{1}{2}\right), \qquad q(x) = \kappa\delta\left(x - \frac{1}{2}\right).$$

Figure 3(a) shows $v(x)$ for the beam supported by a weak spring in the center of the beam. One can see that $v(x)$ is unimodal. If the spring becomes stronger, $\kappa_1 < \kappa/E \leqslant \kappa_2$, but is still located in the center, the maximum of $v(x)$ corresponds to a noncentral applied force. The equally dangerous loadings could be applied in two symmetric eccentric points. The maximum displacement $v(x)$, shown in Figure 3(b), is not a unimodal function of the position of the moving applied force; the design is not optimal. The optimal design for this case (Figure 3(c)) corresponds to two equally stiff springs located symmetrically with respect to the center; the design experiences a bifurcation at the critical value of $\kappa/E = \kappa_1$. An optimally supported beam is shown in Figure 3(c), where two strong springs are located symmetric with respect to the center of the beam. The maximal displacement curve becomes unimodal again, with a large interval of almost constant values in the middle. The next bifurcation occurs when κ further increases, at the point $\kappa/E = \kappa_2$. Three springs appear after the next bifurcation. The number of optimal supporting points increases and tends to infinity when the springs are much stronger than the beam, $\kappa/E \gg 1$. The optimality conditions

$$w'(x_i) = 0, \qquad w(x_i)|_{f=f_i} = \text{constant}(i),$$

give the optimal position of the supporting springs x_i and a requirement on their stiffnesses α_i.

5.2. COMPOSITE STRIP WITH CONSTRAINED DEVIATION OF THE LOADING

This example shows the design of an *optimal structure* for the worst possible loading. Consider an infinite strip $\Omega = \{-\infty < x < \infty, -1 \leqslant y \leqslant 1\}$, made from a two-component elastic composite with arbitrary structure but with fixed fractions m_A and $m_B = 1 - m_A$ of the isotropic components. The stiffness of the composite $C(x, y)$ is an anisotropic elasticity tensor; it is assumed that the stiffness can vary only along the strip, $C = \text{constant}(y)$.

Assume that the upper boundary is loaded by some unknown but uniform loading f,

$$\sigma(x, 1) \cdot N = f \quad \forall x,$$

where $N = (0, 1)$ is the normal vector. The loading f consists of the fixed component $f_0 = (0, 1)$ directed along the normal and a variable component (deviation) (f_N, f_T), the magnitude of the deviation is constrained:

$$f = (f_0 + f_N)N + f_T T, \qquad f_N^2 + f_T^2 = \gamma^2. \tag{41}$$

Here $T = (1, 0)$ is the tangent vector and γ is the intensity of the deviation. The constraint (41) can be rewritten as

$$f = (f_0 + \gamma\cos\theta)N + (\gamma\sin\theta)T \quad \text{for } y = 1,$$

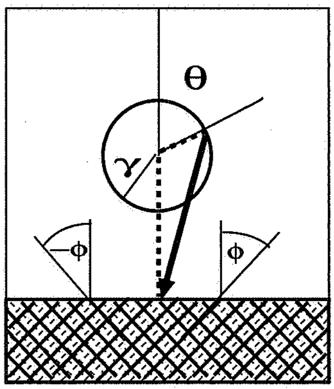


Figure 4. An infinite composite strip loaded by a force f that could deviate from the normal direction. If the norm γ of the deviation is smaller than a critical value γ_1, the optimal composite is a laminate with layers directed across the strip. If γ is greater than γ_1, the optimal composite is second rank laminate with layers oriented along directions ϕ and $-\phi$.

where θ is the angle of inclination of the deviation of the loading; see Figure 4. The lower boundary of the strip is assumed to be loaded by a symmetrically deviated force

$$f_- = -f = -(f_0 + \gamma \cos\theta)N + (\gamma \sin(-\theta))T \quad \text{for } y = -1.$$

The symmetry of the loadings results in the horizontal strain being zero,

$$\epsilon_{xx}(x, y) = 0, \quad -1 \leqslant y \leqslant 1, \tag{42}$$

so that the strain tensor has only two, vertical and shear, nonzero components. The stiffness of the composite $C(x)$ is an anisotropic tensor that is assumed to vary only along the x coordinate. We consider the problem of optimization of the principal compliance of the described domain.

5.2.1. *Design Parameters*

Applying the symmetry theorem, we conclude that:

1. The elastic properties of the optimally designed structure do not vary along the strip, since the design is invariant to the translation $x \to x + \chi$. Together with the assumption that the material properties do not vary with the thickness, this leads to the conclusion that the elastic properties are uniform: the tensor C is constant in x and y. This implies that the stress field σ is constant inside an optimal strip and

$$\sigma_{yy} = 1 + \gamma \cos\theta, \qquad \sigma_{xy} = \gamma \sin\theta. \tag{43}$$

2. The material in the optimal strip is orthotropic with main axes directed along the x and y axes since the design is invariant to the reflection $x \to -x$:

$$C : \begin{pmatrix} 0 & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{pmatrix} = C : \begin{pmatrix} 0 & -\epsilon_{xy} \\ -\epsilon_{xy} & \epsilon_{yy} \end{pmatrix}.$$

This implies orthotropy with the main axes codirected along the x, y axes.

For the following calculations, we introduce an orthonormal $(a_i : a_j = \delta_{ij})$ tensor basis

$$a_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad a_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{44}$$

In this basis, the stress tensor σ,

$$\sigma = \begin{pmatrix} \sigma_2 & \sigma_3 \\ \sigma_3 & \sigma_1 \end{pmatrix},$$

is represented as a vector

$$\sigma = \sigma_1 a_1 + \sigma_2 a_2 + \sqrt{2}\sigma_3 a_3.$$

The compliance tensor S and stiffness tensor $C = S^{-1}$ are presented as matrices with the components $\{S_{ij}\}$ and $\{C_{ij}\}$; their orthotropy implies the representation

$$S = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{33} \end{pmatrix}$$

and a similar one for C.

5.2.2. *The Optimization Problem*

The energy Π of an orthotropic material is computed either as a function of stresses and compliance tensor $S = \{S_{ij}\}$ (stress energy):

$$\Pi_\sigma(S, \sigma) = \frac{1}{2}(S_{11}\sigma_1^2 + S_{22}\sigma_2^2 + 2S_{12}\sigma_1\sigma_2 + 2S_{33}\sigma_3^2), \tag{45}$$

or as a function of strain ϵ and stiffness tensor $C = \{C_{ij}\}$,

$$\Pi_\epsilon(C, \epsilon) = \frac{1}{2}(C_{11}\epsilon_1^2 + C_{22}\epsilon_2^2 + 2C_{12}\epsilon_1\epsilon_2 + 2C_{33}\epsilon_3^2). \tag{46}$$

Recall (see (43)) that two components $\sigma_1 = \sigma_{yy}$ and $\sigma_3 = \sigma_{xy}$ of the stress field σ are known, and the strain in the xx direction is zero, (42):

$$\epsilon_2 = S_{12}\sigma_1 + S_{22}\sigma_2 = 0;$$

therefore, σ_2 can be excluded. The elastic energy (46) becomes

$$\Pi_\epsilon(C, \epsilon) = \frac{1}{2}(C_{11}\epsilon_1^2 + 2C_{33}\epsilon_3^2)$$

or, in terms of stress (see (45)),

$$\Pi_\sigma(S, \sigma) = \frac{1}{2}\left(\left(S_{11} - \frac{S_{12}^2}{S_{22}}\right)\sigma_1^2 + 2S_{33}\sigma_3^2\right).$$

The problem of robust optimal design becomes

$$P_{\text{strip}} = \min_{C \in G_m \text{ closure}} \max_{f \in F} \Pi(S, \sigma), \tag{47}$$

where G_m closure is the set of all possible effective compliance tensors of a microstructure formed from the two given materials with the compliance tensors S_A and S_B, taken in the proportion m_A and $m_B = 1 - m_A$, respectively, see [8, 27]. We reformulate the problem using a sum of weighted energies, where the minimized functional is taken as a sum of the energies due to the extreme loadings.

5.2.3. *Laminates of Third Rank: Symmetry*

The description of the strongest structures that minimize the sum of the energies due to several loadings is known, (see the original papers [2, 3, 17] and the books [8, 29]); the best structures in 2D are so-called "laminates of the third rank" shown in Figure 5. In 3D, they are the sixth rank laminates [17]. Structural optimization based on using the third rank composites was effectively developed for the multi-loadings case in [6, 10, 25]. The effective compliance tensor $S = C^{-1}$ of a third rank composite – the symmetric fourth-order tensor of elasticity – has the representation

$$S = S_A + m_B\big((S_B - S_A)^{-1} + m_A N\big)^{-1}, \tag{48}$$

where S_A is the compliance of an enveloping (reinforcing) material, S_B is the compliance of the material in the nucleus, N is the matrix of structural parameters that depends on the structure of the composite, see [8, 29],

$$N = E_A \sum_{i=1}^{3} \alpha_i P(\phi_i), \qquad \sum_{i=1}^{3} \alpha_i = 1, \quad \alpha_i \geqslant 0.$$

Here E_A is the Young's modulus of the A-material, angles ϕ_i are the angles that define the directions of laminates (directions of reinforcement), P is a tensor product of four directional vectors $z_i = (\cos\phi_i, \sin\phi_i)$:

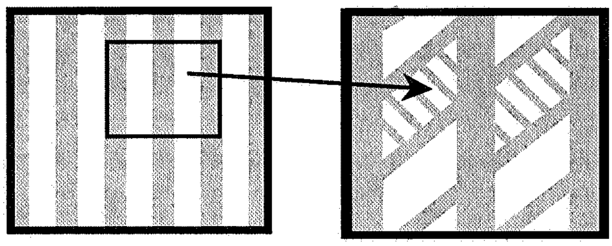


Figure 5. The schematic picture of the composite of the third rank.

$$P(\phi_i) = z_i \otimes z_i \otimes z_i \otimes z_i, \tag{49}$$

α_i is the corresponding relative thickness of the reinforcing layer in the ith direction.

The above mentioned symmetry of an optimal composite requires the orthotropy of the optimal structure. Since the original materials are isotropic, the structure is orthotropic if the matrix N is orthotropic. This can be achieved by setting

$$\phi_2 = -\phi_3 = \phi, \qquad \alpha_2 = \alpha_3 = \alpha.$$

Generally, the optimal strip is reinforced by three layers of strong material; one layer (with relative volume fraction $1 - 2\alpha$) is directed in the y-direction and two other layers (with equal relative volume fractions α) are symmetrically inclined by the angles $\pm\phi$. In addition, the structure may degenerate into a single layer (when $\alpha = 0$) or two symmetric layers (when $\alpha = \frac{1}{2}$) with angles ϕ and $-\phi$. Because of this symmetry, the matrix N for an optimal composite becomes

$$N = (1 - 2\alpha)P(0) + \alpha P(\phi) + \alpha P(-\phi). \tag{50}$$

Let us compute the compliance of a third-rank composite in the basis (44). Compliance S_A of an isotropic material A is given by a matrix

$$S_A = \frac{1+\nu_A}{E_A}\begin{pmatrix} 1-\nu_A & -\nu_A & 0 \\ -\nu_A & 1-\nu_A & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and similarly for the material B. To compute the effective compliance of a third-rank laminate, we first represent the matrix $P(\phi)$ of (49) in the basis (44),

$$P(\phi) = \begin{pmatrix} \cos^4\phi & \sin^2\phi\cos^2\phi & \sqrt{2}\sin\phi\cos^3\phi \\ \sin^2\phi\cos^2\phi & \sin^4\phi & \sqrt{2}\sin^3\phi\cos\phi \\ \sqrt{2}\sin\phi\cos^3\phi & \sqrt{2}\sin^3\phi\cos\phi & 2\sin^2\phi\cos^2\phi \end{pmatrix},$$

and obtain from (50)

$$N = \begin{pmatrix} 1 - 2\alpha + 2\alpha\cos^4\phi & 2\alpha\sin^2\phi\cos^2\phi & 0 \\ 2\alpha\sin^2\phi\cos^2\phi & 2\alpha\sin^4\phi & 0 \\ 0 & 0 & 4\alpha\sin^2\phi\cos^2\phi \end{pmatrix}.$$

The matrix N is the variable part of the compliance matrix, (see (48)); it depends on only two scalar parameters, ϕ and α.

The structural optimization problem (47) finally becomes an algebraic problem

$$J_{\text{strip}} = \min_{\phi,\alpha}\max_{\theta}\Pi_\sigma(S(\phi,\alpha),\sigma(\theta)); \tag{51}$$

the expressions for the quantities involved are described above. The angle θ is the angle of deviation of the loading from the normal, and ϕ and α are structural parameters.

5.2.4. *Second Rank Structure is Optimal*

Although in the general case of minimization of a sum of energies corresponding to multiple loadings the third-rank laminates are optimal, here the optimal structures are the second – not the third-rank laminates. To prove this statement we must find the derivative of Π_σ in the algebraic minimization problem (51), and demonstrate that it does not become zero; this would give the optimal value of α on the boundary of the constraint. However, we skip this bulky calculation and give a physical argument supported by results of numerical optimization. Because of the absence of a displacement in the x-direction, there is no need to reinforce this direction. Even more, the stress in the composite does not change if a layer with infinite stiffness oriented along x-axes is added to the composition. If this infinitely stiff layer is counted, then the structure would be reinforced by three layers of stiff material. Since the stiffness of a structure with an infinitely stiff layer is not smaller than the stiffness of a structure without such a layer, the optimality of the second-rank laminates follows.

This conclusion is supported by results of numerical optimization, which gives $\alpha_{\text{opt}} = 1/2$ for all settings. Physically, this means that the optimal structure is reinforced by either single laminates oriented across the strip (the case when $\phi = 0$) or by a second-rank laminate with two symmetric reinforcement directions ϕ and $-\phi$, see Figure 4. This degeneration of the third-rank laminates can be explained by the special geometry of the strip and the loading, which do not allow for any strain ϵ_{xx} along the strip, and the assumed independence of the design on the y-coordinate. The formulas for the effective properties of a symmetric second-rank composite are simplified: They are still given by the expression (48) but the structural matrix N is

$$N = \frac{1}{2}(P(\phi) + P(-\phi))$$

instead of (50); in the basis (44) it has the form

$$N = \begin{pmatrix} \cos^4\phi & \sin^2\phi\cos^2\phi & 0 \\ \sin^2\phi\cos^2\phi & \sin^4\phi & 0 \\ 0 & 0 & 2\sin^2\phi\cos^2\phi \end{pmatrix}.$$

We notice that the symmetry in this example efficiently reduces the dimension of the computational problem, but the general method works with or without symmetry.

5.2.5. *Numerical Example*

For the first example, the following values of parameters were chosen:

$$m_A = 1 - m_B = 0.2, \qquad E_A = 1, \qquad E_B = 5,$$
$$\nu_1 = \nu_2 = 0.3, \qquad f_0 = 1.$$

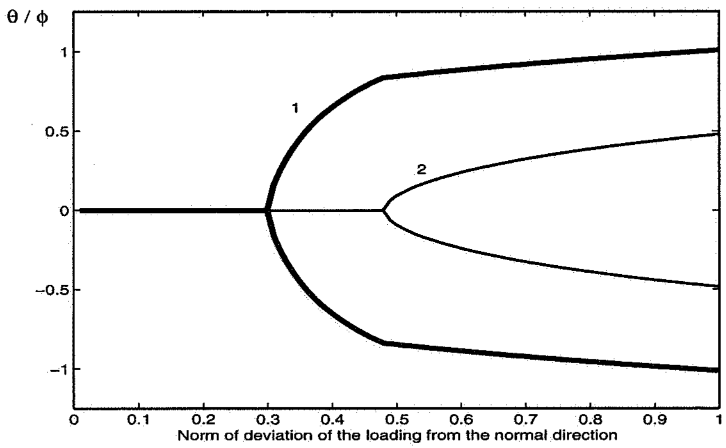


Figure 6. Bifurcation diagram shows (1) the angle of deviation $\hat{\theta}(\gamma)$ of the extreme, most dangerous loading and (2) the angle $\hat{\phi}(\gamma)$ of optimal reinforcement of the second rank laminated composite. Notice that the bifurcation parameter γ has different critical values for the deviation of the loading θ and for the angle of reinforcement ϕ.

The relative magnitude γ of the variable part of the loading is the parameter of the problem; the angle θ of the optimal deviation of the extreme loading and the structural parameters α and ϕ are determined from the solution of the min–max optimization problem. We detect three regimes:

1. When $\gamma < \gamma_0 = 0.31$, the extreme loading is vertical, $\theta_{\text{opt}} = 0$, and the optimal structure is a laminate with vertical layers directed across the strip, $\phi_{\text{opt}} = 0$, see Figure 6.
2. At the critical value γ_0 of the parameter γ, the direction of the extreme deviation undergoes a bifurcation, $\theta_{\text{opt}} = \pm\hat{\theta}(\gamma)$, shown by the curve 1 in Figure 6. But for $\gamma < \gamma_1 = 0.46$, the optimal structure remains the same: a laminate with layers directed across the strip, $\phi_{\text{opt}} = 0$ (curve 2 in Figure 6).
3. When the magnitude γ further increases, $\gamma \geqslant \gamma_1$, the optimal structure bifurcates as well; it becomes a second-rank matrix laminate with the angle $\phi_{\text{opt}} = \pm\hat{\phi}(\gamma)$ (curve 2 in Figure 6).

Although the problem has two solutions for the extreme loading, the dependence of the compliance on the parameters ϕ and θ is a saddle-point surface as is shown in Figure 7. Indeed, the problem is reformulated (relaxed) accounting for non-uniqueness of the loading and for the symmetry in the design.

The following examples demonstrate the dependence of the optimal solution on the ratio of Young's moduli for the materials in the composite. Figure 8 shows the bifurcation diagrams for different ratios of Young's moduli. Qualitatively, the picture remains the same, but the critical values of the bifurcation parameter γ are different: The larger the ratio, the smaller the critical value of γ_0 and γ_1 at which the bifurcation occurs. The interval (γ_0, γ_1) decreases with an increase of the ratio of Young's moduli.

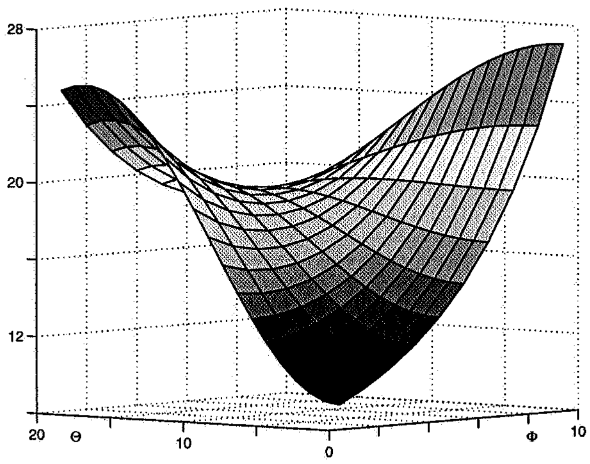


Figure 7. Energy stored in the composite is a saddle point function of the angle of deviation of the loading θ and of the direction of reinforcement ϕ.

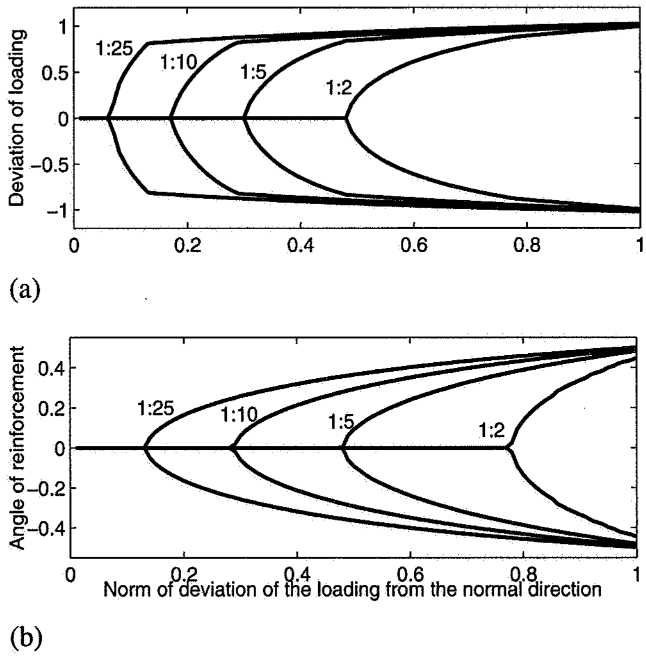


Figure 8. Bifurcation diagram for different ratios of Young's moduli of the materials in the composite ranging from 1 : 2 to 1 : 25. (a) Bifurcation of the angle $\hat{\theta}(\gamma)$ of deviation of the extreme loading from the normal. (b) Bifurcation of the angle $\hat{\phi}(\gamma)$ of direction of the optimal reinforcement for the second rank laminated composite.

5.3. DISCUSSION

The principal compliance is a basic characteristic of an elastic body which depends only on the shape of the domain and on the stiffness of the material. By the proper normalization of Λ using $\|\Omega\|$ and $\|C\|$, this quantity is reduced to the dimensionless parameter λ:

$$\lambda = \frac{\Lambda}{\|\Omega\|\ \|C\|},$$

and can be treated as a basic integral characteristic of the filled domain along with such properties as main eigenfrequency, the capacity, etc.

The optimal design aimed to decrease the principal compliance is a minimax problem; typically, the problem does not have a saddle point and the optimal design provides equal minimal compliance for several extreme loadings. Symmetries and relaxation bring the problem to a saddle-point type. Depending on the type of constraints, the extreme loading can be a principal eigenfunction of an eigenvalue problem, a concentrated loading, or a solution of a bifurcation problem.

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