Optimal Source Control and Resolution in Nondestructive Testing

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Abstract

The paper considers a problem of damage detection arising in nondestructive testing. A conclusion about a presence of damage inside the body should be made from measuring boundary responses due to applied boundary currents and comparing them with the responses of the undamaged sample. The damage can be detected if there exist such a current in the set of the currents prescribed by the conditions of the experiment, which generates perturbation in the boundary measurements greater than the given noise level.

Using variational approach the detection problem is formulated as a generalized eigenvalue problem. The maximal eigenvalue determines the accuracy of the measurements needed to detect this distribution of damaged material. The resolution problem for a class of defects is formulated as a problem of minimization of the maximal eigenvalue over a set of functions characterizing damage.

To consider the worst case of detection the damaged material should be distributed throughout the body in order to minimize the maximal eigenvalue of the spectral operator. An analytical estimate of the perturbation of the maximal eigenvalue is given depending on the amount of damaged material.

Key words: nondestructive testing, generalized eigenvalue problem, resolution, Neumann to Dirichlet operator, inexact measurements.

1 Introduction

The present paper considers a problem of detection of damaged parts inside a body when damaged material is characterized by different properties in comparison with undamaged sample. Applying currents on the boundary and measuring the corresponding responses also on the boundary (or on the part of the boundary) a conclusion should be made about a presence of inclusions of damaged material inside the body when the measurements are of limited precision.

Two problems are formulated:

(1) The problem of optimization of boundary sources - finding an excitation from a given set of functions which maximizes the measured difference in the responses. This problem is a problem of optimal boundary control and is formulated using variational approach as a generalized eigenvalue problem. The maximal eigenvalue gives an estimate for the accuracy of the measurements needed to detect this particular distribution of damaged material inside the body.

(2) The problem of assured detection - finding an inclusion which is the most hidden or the most difficult to detect. This gives an estimate for a necessary accuracy of the measurements in the worst possible case of distribution of the damage. This second problem is a problem of minimization of the maximal eigenvalue, when the control is the property of the material in the interior of the body. Analytical solution for this problem is constructed based on the results of the paper [1] for the case when the sample is made of homogeneous material and the damaged material differs by its conductivity from the material of the sample. An analytical estimate is given of the total amount of damaged material inside a homogeneous sample which is possible to detect with measurements of a given accuracy.

2 Variational formulation of the problem of detection in nondestructive testing

The problem of detection is a problem of identification of the presence of a defect inside a body in a situation when measurements are of limited precision. It is assumed that currents can be applied on the boundary of the real body and on the boundary of the undamaged (or standard) sample. The measured are voltages or some other functional linearly dependent on the applied sources. The currents can be chosen from some set of functions describing physical constraints of the experiment, for example, that the energy of the applied source is limited or the electrodes are placed only on a part of the boundary.

If the difference between the measured and standard values of the functional is greater than the level of the noise in the measurements, which is assumed to be known, than the real sample certainly contains some defects inside. If the responses differ by less than possible noise in data, we can say nothing about a possible presence of an inclusion.

The optimization problem is to choose excitations from a given set of functions which would maximize the measured difference in responses, and to determine what damage inside the sample is possible to detect having measurements with a prescribed level of noise η.

Let the body contain some parts of a damaged material inside, so that the change of the conductivity is described by the function σ. We assume that we can inject a current f, chosen from some set of functions $f \in \mathcal{F}$ described by the constraints of the experiment

$$\mathcal{F} = \{ f : \langle Hf, f \rangle = 1 \}, \tag{1}$$

with selfadjoint operator H, and $\langle \cdot, \cdot \rangle$ be a scalar product in an appropriate space of functions. We measure a difference in the responses given by a function g which is the value of a (linear with f) functional A_σ depending on the defect σ for the applied current f. A_σ is assumed to be selfadjoint.

The optimization problem for the optimal current is to find a function f in the set $\mathcal{F}$ which maximizes the norm of the measured function g. It can be written as a variational problem of maximization of a ratio of energies, which has the form of Rayleigh quotient functional:

$$J = \max_{f \in \mathcal{F}} \langle A_\sigma f, f \rangle = \max_f \frac{\langle A_\sigma f, f \rangle}{\langle Hf, f \rangle} \tag{2}$$

When A_σ is the difference in the voltage measurements due to the same injected current, $A_\sigma = V_\sigma$, then $\langle V_\sigma f, f \rangle$ is the difference of the energies spent for injection of the same current f into the real and the standard samples. In [1] the functional (2) was considered with such operator A_σ and

H mapping the injected currents into the measured voltages in the standard sample, which restricts the energy of the current dissipated in the standard sample. For $A_\sigma = V_\sigma^* V_\sigma$ the numerator of the functional J is a squared norm of the measured difference between the voltages in the real and the standard samples. With H equal identity, the norm of the applied currents is assumed to be unit; for this particular case the spectral problem of this type was formulated in [6, 5]. Assuming H to be an operator mapping injected currents into the measured voltages in the real body, we restrict the energy of the injected current to be unit.

Suppose, that the maximal value of this functional is λ_0. Then varying with respect to f we obtain the generalized eigenvalue problem with the eigenvalue λ_0:

$$A_\sigma f = \lambda_0 \, Hf \ . \tag{3}$$

The eigenvalue λ_0 defines the accuracy of the measurements which is necessary for the detection of the defect σ using this type of experiment. Noise level η in measurements should be less than λ_0,

$$\lambda_0 > \eta \ , \tag{4}$$

in this case the first eigenfunction of (3) provides difference in measurements on the boundary greater than the noise level, and the presence of the defects can be identified.

If the currents can be injected only on a part of the boundary $\partial\Omega_1$ and the voltages are to be measured on the part $\partial\Omega_2$, then to account this the scalar products in (2) and (3) can be taken as $\langle \, Af, \, f \, \rangle_{\partial\Omega_2}$ and $\langle \, Hf, \, f \, \rangle_{\partial\Omega_1}$.

For each particular application the problem can be specified to account constraints of the experiment. We consider bellow two particular cases of the detectability problem (2) - (3) and derive corresponding generalized eigenvalue problems: for a case when the electrodes can be placed only on a part of the boundary Γ, $\Gamma \subset \partial\Omega$, and for a case when in order to detect a damage we compare the energies spent for injecting the same current into the real body and into the standard sample.

3 Spectral problem with boundary eigenfunctions

Let a body Ω with the boundary $\partial\Omega$ be filled with an isotropic material of the conductivity γ. We assume that arbitrary currents f can be applied on the surface of the body and the corresponding responses are measured. The correspondence between the applied currents and the measured voltage responses is given by a linear operator $\mathcal{L}_\gamma$ depending on the conductivity γ:

$$\nabla \cdot \gamma \nabla u = 0,\ x \in \Omega, \quad \gamma \frac{\partial u}{\partial n} = f,\ x \in \partial\Omega\ , \tag{5}$$

where the current f satisfies the integral restriction: $\int_{\partial\Omega} f\ dx = 0$. The solution of equation (5) is unique up to a constant component which is determined if a zero value of the function is prescribed; so we assume that $\int_{\partial\Omega} u\ dx = 0$.

Boundary voltages are given by the Neumann to Dirichlet operator R_γ, $L_2(\partial\Omega) \to L_2(\partial\Omega)$, which put the correspondence between the injected currents and the measured voltages.

$$R_\gamma f = u \ \text{ on } \ \partial\Omega\ , \ \text{ such that } \ f = \gamma \frac{\partial u}{\partial n} \tag{6}$$

This mapping is inverse to the Dirichlet to Neumann boundary map Λ_γ which transforms the potential on the boundary to the values of the normal derivative [8].

Let the body contain some inclusions of damaged material inside so that its conductivity is described by the function γ^*. Assuming that the perturbation of the properties $\sigma = \gamma^* - \gamma$ is small, consider operator $\mathcal{L}_{\gamma^*}$ which is a perturbed operator $\mathcal{L}_\gamma$ (5). The function γ^* describes the conductivity of the real medium with inclusions of damaged material, meanwhile the known unperturbed function γ is the conductivity of the model medium.

Let an operator V_σ be the difference in the boundary voltage measurements generated by the current f, corresponding to the media with the conductivities γ^* and γ:

$$V_\sigma f = R_{\gamma^*} f - R_\gamma f\ , \tag{7}$$

with R_γ defined in (6). Its norm is given by the first singular value μ_1:

$$\mu_1 = \max_{\|f\|=1} \| V_\sigma f \|\ . \tag{8}$$

The problem with data only on a part of the boundary. As an example we consider a case when the injecting and measuring electrodes can be placed only on a part of the boundary Γ, $\Gamma \subset \partial\Omega$. Using the developed approach the detectability problem can be formulated as a generalized eigenvalue problem with the eigenvalues in mixed boundary condition.

The optimal currents injected on the part of the boundary Γ which maximize the difference in voltage responses $u = R_\gamma f$ and $w = R_{\gamma^*} f$ due to the media of conductivity γ and γ^* solve the following optimization problem

$$J = \int_\Gamma (R_{\gamma^*} f - R_\gamma f)^2 \, dx \to \max_{f:\, f \in \mathcal{F}_0} , \tag{9}$$

where we assume that $\mathcal{F}_0 = \{\|f \chi_\Gamma\|_{L_2(\partial\Omega)} = 1\}$. Here χ_Γ is the index function of the domain Γ:

$$\chi_\Gamma(x) \;=\; \begin{cases} 1, & \text{if } x \in \Gamma \\ 0, & \text{if } x \in \partial\Omega \setminus \Gamma \end{cases} . \tag{10}$$

The problem (9) is homogeneous, let its maximal value be μ. Varying it and taking into account the differential constraints we obtain a system of Euler Lagrange equations for the optimal potentials u and w and the Lagrange multipliers p and q. The system of equations for the optimal u and w contains the Lagrange multipliers p and q in the mixed boundary condition:

$$\begin{aligned} \nabla \cdot \gamma \nabla u = 0,\ x \in \Omega, &\quad (p+q)\,\chi_\Gamma - 2\,\mu\,\gamma \frac{\partial u}{\partial n} = 0,\ x \in \partial\Omega, \\ \nabla \cdot (\gamma + \sigma) \nabla w = 0 \ \ x \in \Omega, &\quad (p+q)\,\chi_\Gamma - 2\,\mu\,\gamma \frac{\partial w}{\partial n} = 0,\ x \in \partial\Omega. \end{aligned} \tag{11}$$

The system of the equations for the Lagrange multipliers p and q contains the functions u and w in the boundary condition:

$$\begin{aligned} \nabla \cdot \gamma \nabla p = 0,\ x \in \Omega, &\quad 2\,(w-u)\,\chi_\Gamma + \gamma \frac{\partial p}{\partial n} = 0,\ x \in \partial\Omega, \\ \nabla \cdot (\gamma + \sigma) \nabla q = 0,\ x \in \Omega, &\quad 2\,(w-u)\,\chi_\Gamma - \gamma \frac{\partial q}{\partial n} = 0,\ x \in \partial\Omega. \end{aligned} \tag{12}$$

The solution of the system of equations (11) - (12) is provided by a system of functions

$$\{u,\, w,\, -2\lambda u,\, 2\lambda w\}, \quad \text{with the eigenvalue } \lambda : \ \lambda^2 = \mu. \tag{13}$$

Substituting this solution into the system (11) we obtain a system of Euler equations for the optimal functions solving the problem (9) in a form of a generalized eigenvalue problem:

$$\nabla \cdot \gamma \nabla u = 0, \; x \in \Omega, \quad (w-u)\, \chi_\Gamma - \lambda \gamma \frac{\partial u}{\partial n} = 0, \; x \in \partial\Omega,$$
$$\nabla \cdot (\gamma+\sigma) \nabla w = 0, \; x \in \Omega, \quad (w-u)\, \chi_\Gamma - \lambda \gamma \frac{\partial w}{\partial n} = 0, \; x \in \partial\Omega. \tag{14}$$

This is a generalized eigenvalue problem for the functions u and w with the eigenvalue λ in the mixed boundary condition. The absolute value of the first eigenvalue λ_1 equals to the norm μ_1 of the operator V_σ: $|\lambda_1| = \mu_1$ if the same constraint, that the applied currents f are different from zero only on Γ, is used in (5).

The corresponding variational functional has the form of Rayleigh quotient functional:

$$\lambda \; = \; \frac{\int_\Gamma (w-u)^2 dx}{\int_\Omega (\gamma+\sigma) \nabla w \cdot \nabla w \, dx - \; \int_\Omega \gamma \nabla u \cdot \nabla u \, dx} \; . \tag{15}$$

In what follows we want to estimate the maximal eigenvalue or the norm of the operator V_σ. It naturally arises in the problem of nondestructive testing or in the inverse boundary value problem ([8, 5, 4, 2]). A variant of the inverse problem is the problem of detectability of a small inclusion in a case when measurements are inexact.

4 On minimization of the maximal eigenvalue and the resolution problems

Let us consider the first eigenvalue μ_1 of the measured voltage difference V_σ as a function of the perturbation $\sigma = \gamma^* - \gamma$ from some class of inclusions Σ: $\mu_1 = \mu_1(\sigma)$.

All inclusions from Σ which can be detected by boundary measurements with the noise level η form a set of perturbations of conductivity $\mathcal{S}_\gamma(\eta)$:

$$\mathcal{S}_\gamma(\eta) = \{\sigma \in \Sigma : \quad \mu_1(\sigma) \; > \; \eta\}. \tag{16}$$

The problem whether all inclusions σ from a class Σ can be detected is a problem of estimating a lower bound with respect to σ of the maximal eigenvalue of the disturbance operator:

$$\min_{\sigma\in\Sigma} \mu_1(\sigma). \tag{17}$$

If this minimum greater than the noise level η than any inclusion from the class Σ can be detected. The function σ which provides minimum is the inclusion or perturbation which is the most difficult for detection. Solution of the problem (17) gives an estimate for the accuracy of the measurements, which is needed to detect any inclusion from the class Σ.

Parametrization of the set of inclusions. Let us parametrize somehow the set of inclusions Σ. Let, say, functions σ from Σ are characterized by the parameters $p_1, ..., p_n$ changing in some n-dimensional set P_0. Then the resolution problem for inclusions in the class Σ can be written as

$$\min_{(p_1,...,p_n)\in P_0} \mu_1(p_1, ..., p_n) \geq \eta. \tag{18}$$

If the noise level η satisfies (18) then all the parameters P_0 can be resolved from the measurements.

Let a set P_1 be a subset of the set P_0. It is easy to show that for inclusions characterized by parameters from the set P_1

$$\min_{(p_1,...,p_n)\in P_1} \mu_1(p_1, ..., p_n) \geq \min_{(p_1,...,p_n)\in P_0} \mu_1(p_1, ..., p_n), \quad \text{if} \quad P_1 \subset P_0. \tag{19}$$

Therefore if the parameters from the set P_0 can be resolved from the boundary from the measurements of a given noise level, then certainly this is true for the parameters from the set P_1.

This observation permits to use this approach in order to construct the resolution curves. In [3] the resolution curves are constructed for a class of inclusions Σ described by a set of trial balls $B(l, h)$ of diameter l located at a distance h from the boundary,

$$\sigma(x) \;=\; \begin{cases} 1, & \text{if } x \in B(l,h) \\ 0, & \text{otherwise} \end{cases} \quad . \tag{20}$$

Then $\mu_1(\sigma)$ as a function of the parameters l and h forms a resolution function: all inclusions with $\mu_1(l, h) > \eta$ form a class $\mathcal{S}_\gamma(\eta)$, they are detectable or resolvable from the boundary measurements. All inclusions with

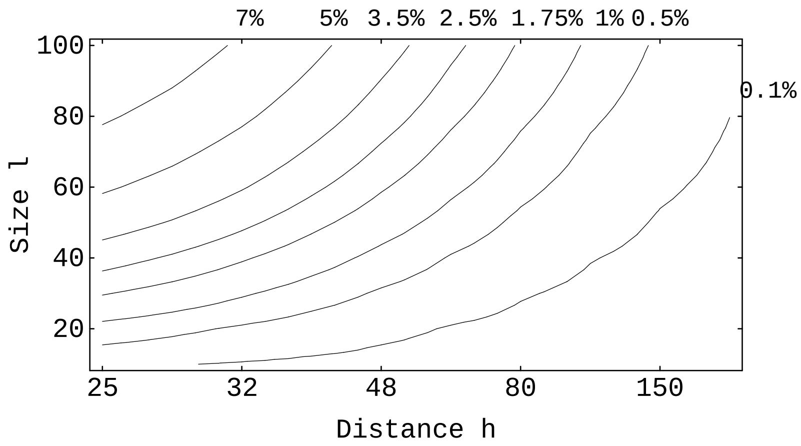

Figure 1: The normalized resolution curves for a detection of a localized inclusion of the size l located at the distance h from the side bearing electrodes.

$\mu_1(l,h) \leq \eta$ are not detectable by the boundary measurements of the given accuracy. Hence the contour lines of the function $\mu_1(l,h)$ on the plane (l,h) divide the complementary subsets of the perturbations σ which generate respectively detectable and undetectable difference in measurements on the boundary.

Using described in [3] numerical approach the contour lines of the function $\mu_1(l,h)$ are shown on Figure 1 for the case when the electrodes are placed on one side of the sample. These are the resolution curves for detection of a localized inclusion of resistivity 1 ohm-m in a standard sample of resistivity 100 ohm-m. The curves are normalized to the unit norm of the response in a standard injection, therefore they show the accuracy of the measurements η in terms of percentage of the noise to the useful signal.

5 An estimate for the resolution depending on an amount of damaged material

Here we estimate an accuracy of measurements needed to detect a presense of a defect as a function of the total amount of damaged material.

Let the class Σ consist of the functions σ with fixed total volume of damaged material equal m_0. Let the conductivity of the model sample γ

be constant, and the damaged material be of conductivity $\gamma_1, \gamma_1 < \gamma$. The conductivity of the real sample is

$$\gamma^*(x) = \gamma_1 \ \chi(x) + \gamma \ (1 - \chi(x)) \ , \tag{21}$$

where $\chi(x)$ is the index function of a domain Ω_ϵ occupied by the inclusions of damaged material:

$$\chi(x) \ = \left\{ \begin{array}{ll} 1, & \text{if } x \in \Omega_\epsilon \\ 0, & \text{if } x \in \Omega \setminus \Omega_\epsilon \end{array} \right. . \tag{22}$$

Let the set of all possible functions $\chi(x)$ be restricted by restricting the total volume of the damage:

$$\Sigma_\chi = \{ \ \chi(x) : \ \ \int_\Omega \chi(x) \, dx = m_0 \ \}. \tag{23}$$

Let us assume the following scheme of an experiment: currents are injected in the real sample and the standard body, and the difference between the dissipated energies in both injections is measured. If this difference is greater than a noise level in measurements, then the real sample contains some defects. Following developed approach to the resolution problem, let us consider a detectability functional D as a difference in energies of the currents dissipated in the real sample and the standard body:

$$D = \int_{\partial\Omega} (R_{\gamma^*} f - R_\gamma f) \, f \, dx, \tag{24}$$

for currents f with unit energy spent in the standard injection:

$$\mathcal{F} = \{ f : \ \ \int_{\partial\Omega} f \, R_\gamma f \, dx = 1 \}. \tag{25}$$

The optimal currents f in such an experiment are those maximizing the value of this functional:

$$D \to \max_{f \in \mathcal{F}} . \tag{26}$$

Varying this functional with the differential constraints we obtain again a system of Euler equations for the optimal currents of the type (14) but with a different boundary condition:

$$\lambda \, u - (w - u) = 0 \ \text{ on } \ \partial\Omega \ . \tag{27}$$

Again we deal with a generalized eigenvalue problem with the eigenvalue λ in the boundary condition. From the assumption that the damaged parts are less conductive follows that the first eigenvalue λ_1 is positive. For more conductive damage a squared value of the functional should be considered in (26).

The variational functional has the form:

$$\lambda \;=\; \frac{\int_\Omega \gamma \nabla u \cdot \nabla u \, dx \;-\; \int_\Omega (\gamma + \sigma) \nabla w \cdot \nabla w \, dx}{\int_\Omega \gamma \nabla u \cdot \nabla u \, dx} \,. \tag{28}$$

Hence the first eigenvalue λ_1 equals to the maximal difference in energies dissipated in the bodies with and without defects.

The problem is to find the most hidden inclusion, which minimizes the eigenvalue λ_1 as a function of the damage.

$$\min_{\chi \in \Sigma_\chi} \lambda_1 (\chi) = \min_{\chi \in \Sigma_\chi} \max_{f \in \mathcal{F}} D(\chi, f) \tag{29}$$

This problem is a problem of minimization of the maximal eigenvalue using a control χ.

The problem (29) can be considered as a min-max game with the functional D, when the player wishing to maximize the value of the functional chooses strategies f from the set $\mathcal{F}$ (25), while the other player minimizes D using strategies χ from the set Σ_χ (23).

It is shown in [1] that the solution of the problem (29) is given by the inclusions of damaged material dispersed into infinitely small pieces and distributed uniformly throughout the body; hence the optimal inclusions form a composite material. The inclusions form a second rank laminated isotropic effective medium ([7]); the optimal conductivity is

$$\gamma_* = \gamma \;+\; m_0 \;\; \left(\frac{1}{\gamma_1 - \gamma} \;+\; \frac{1 - m_0}{n \, \gamma_1}\right)^{-1} , \tag{30}$$

where n is dimension. The optimal currents generate homogeneous and constant currents distribution inside the body.

Let κ denote a ratio of the conductivity difference to the model conductivity, $\kappa = (\gamma - \gamma_1)/\gamma$. The minimum value d_0 of detectability is calculated as:

$$d_0 = \min_\chi \lambda_1 = \frac{m_0 \, \kappa}{1 - m_0 \, \kappa - (1 - m_0) \, \kappa \, n^{-1}} \,. \tag{31}$$

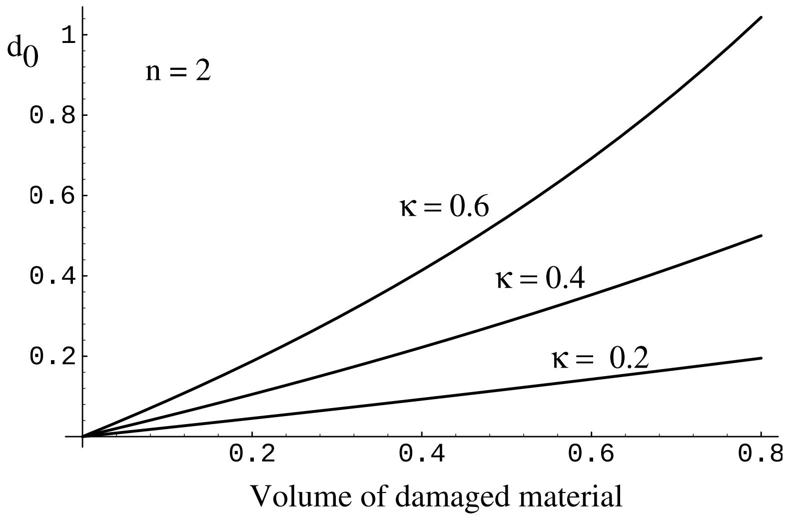

Figure 2: Dependence of the necessary accuracy of the measurements η on the volume fraction of the damaged material for 2-dimensional sample.

Figure 2 shows how the necessary accuracy of the measurements depends on the volume fraction of the damaged material for the relationship between the conductivities $\kappa = 0.2, 0.4, 0.6$.

Hence the value of the functional (26) for the damaged material of the total volume m_0 spread out through all the body, gives the lower estimate for the detectability and defines an accuracy of the measurements needed to detect this amount of damage in the worst possible case. It means that any other distribution of the damaged material can be certainly detected using these measurements.

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