but you could never see the difference in Fig. 3.3. There is a boundary layer at each end in which all the action occurs. The layer reaches approximately to \( x = 8\sqrt{c} \), which is enough for the special solution \( u = 1 - e^{-x/\sqrt{c}} \) to climb from \( u(0) = 0 \) to \( u = 1 - e^{-8} \). At that point it has virtually met the interior solution \( U = 1 \). Then a similar exponential at the other end connects \( U = 1 \) back to \( u = 0 \), in another boundary layer.

The perturbation is singular because the unperturbed solution \( U = 1 \) completely misses the boundary conditions. The leading term \(-cu''\) is disappearing as \( c \) goes to zero, but it remains powerful inside the layer. Elsewhere the problem is calm.

\[
U = \frac{1}{2}(x - x^2)
\]

(a) small \( q \)

\[
U = 1
\]

(b) small \( c \)

Fig. 3.3. A regular perturbation and a singular perturbation: \(-cu'' + qu = 1\).

Note finally that a first derivative \( du/dx \) standing alone in (13) would have destroyed the whole framework. It corresponds to adding a skew-symmetric matrix to the existing \( A^TCA \). Such a term does appear in fluid dynamics, and it illustrates the difference between diffusion and convection. Diffusion is symmetric and convection is not.

**EXERCISES**

3.1.1 For a bar with constant \( c \) but with decreasing \( f = 1 - x \), find \( w(x) \) and \( u(x) \) as in equations (8–10).

3.1.2 For a hanging bar with constant \( f \) but weakening elasticity \( c(x) = 1 - x \), find the displacement \( u(x) \). The first step \( w = (1 - x)f \) is the same as in (9), but there will be stretching even at \( x = 1 \) where there is no force. (The condition is \( w = c\, du/dx = 0 \) at the free end, and \( c = 0 \) allows \( du/dx \neq 0 \).)

3.1.3 Suppose a bar is free at both ends: \( w(0) = w(1) = 0 \). This allows rigid motion. Show that if \( u(x) \) satisfies the differential equation and these boundary conditions, so does \( u(x) + C \) for any constant \( C \).

3.1.4 With the bar still free at both ends, what is the condition on the external force \( f \) in order that \(-\frac{dw}{dx} = f(x), w(0) = w(1) = 0 \) has a solution? (Integrate both sides of the equation from 0 to 1.) This corresponds in the discrete case to solving \( A_0^T y = f \); there is no solution for most \( f \), because the left sides of the equations add to zero.
3.1.5 Find the displacement for an exponential force, \(-u'' = e^x\) with \(u(0) = u(1) = 0\).

Note that \(A + Bx\) is the general solution to \(-u'' = 0\); it can be added to any particular solution for the given \(f\), and \(A\) and \(B\) can be adjusted to fit the boundary conditions.

3.1.6 Suppose the force \(f\) is constant but the elastic constant \(c\) jumps from \(c = 1\) for \(x \leq \frac{1}{2}\) to \(c = 2\) for \(x > \frac{1}{2}\). Solve \(-dw/dx = f\) with \(w(1) = 0\) as before, and then solve \(c\, du/dx = w\) with \(u(0) = 0\). Even if \(c\) jumps, the combination \(w = c\, du/dx\) remains smooth.

3.1.7 Find the next term \(W(x)\) in \(u = \frac{1}{2}(x - x^2) + q(\frac{1}{12}x^3 - \frac{1}{24}x^4 - \frac{1}{24}x) + q^2W + \ldots\). Choose \(W\) to match the \(q^2\) terms in \(-u'' + qu = 1\) and to satisfy \(W(0) = W(1) = 0\).

3.1.8 For the negative value \(q = -1\) show that \(u = d_1 \cos x + d_2 \sin x - 1\) satisfies the differential equation \(-u'' - u = 1\). The exponentials are \(e^{ix}\) and \(e^{-ix}\), and they can be replaced by the sine and cosine.

3.1.9 If the condition at \(x = 1\) were \(u'(1) = 0\), why would no boundary layer be needed in Figure 3.3?

3.1.10 Verify that \(u = d_1 e^{x/\sqrt{c}} + d_2 e^{-x/\sqrt{c}} + 1\) is an exact solution to \(-cu'' + u = 1\). The condition \(u = 0\) at \(x = 0\) gives \(d_1 + d_2 + 1 = 0\); find a similar equation from \(u(1) = 0\) and solve for \(d_2\). We expect \(d_2 \approx -1\) to produce the boundary layer at \(x = 0\).

3.1.11 What is the general solution to the constant-coefficient equation \(-u'' + pu' = 0\)? Try exponentials \(u = e^{px}\).

3.1.12 For \(-u'' + pu' = 1\) with small \(p\), find the regular perturbation \(pV\) by substituting \(u = \frac{1}{2}(x - x^2) + pV\) and keeping the terms that are linear in \(p\).

3.1.13 The solution to \(-cu'' + u' = 1\) is \(u = d_1 + d_2 e^{x/c} + x\). Find \(d_1\) and \(d_2\) if \(u(0) = u(1) = 0\), and find their limits as \(c \to 0\). The limit of \(u\) should satisfy \(U' = 1\); which boundary condition does it keep and which end has a boundary layer?

3.1.14 Find the exponentials \(u = e^{px}\) that satisfy \(-u'' + 5u' - 4u = 0\) and the combination that has \(u(0) = 4\) and \(u(1) = 4e\).

3.1.15 Solve the equation \(-u'' = f\) with \(u(0) = 0\) and \(u'(1) = 0\) when \(f\) is a delta function at \(x = \frac{1}{2}\). The impulse \(f\) is zero (and \(u\) is linear, \(u = Ax + B\)) except at \(\frac{1}{2}\), where \(u\) has a unit step down. The bar is stretched above \(x = \frac{1}{2}\); then free.

3.1.16 Solve the same problem with \(u(0) = u(1) = 0\), leading to the Green's function of page 351. The solution to \(-u'' = \delta\) is again piecewise linear.

3.1.17 My class thinks that \(w\) in equation (9) should be \(\int_0^\infty f(x)\, dx + C\). But what constant of integration makes \(w(1) = 0\)?

Notes on the Dirac delta function (\(\delta\) = unit impulse at \(x = 0\))

Its integral from \(-\infty\) to \(x\) is a step function: jump from 0 to 1 at \(x = 0\)

Second integral is a ramp function (= \(x\) for \(x > 0\); solution to \(u'' = \delta\))

Third integral is a quadratic spline (= \(\frac{1}{2}x^2\) for \(x > 0\); jump in second derivative)

Fourth integral is a cubic spline (= \(\frac{1}{2}x^3\) for \(x > 0\); solution to \(u''' = \delta\), p. 177)

Its derivative \(\delta'\) is a doublet (p. 327)

Delta function \(\delta(x)\delta(y)\) in two dimensions: \(\iint f(x, y)\, \delta(x)\delta(y)\, dx\, dy = f(0, 0)\)

Defining property: \(\int u(x)\, \delta dx = u(0)\) for every smooth function \(u\)