Math 5710, Problem Set 1

1. Determine which of the following collections of “vectors” are linearly independent:
   (a) \([0, 3, -1], [2, 1, 0], [0, 0, 1]\).
   (b) \(f_1(x) = x^2 - 1, f_2(x) = 2(x - 1)^2, f_3(x) = x - 1\).
   (c) \(x_1 = (1, 1, 1, \ldots), x_2 = (0, 1, 1, \ldots), x_3 = (0, 0, 1, 1, \ldots)\).

2. Let \(P^n = \text{span}\{1, x, x^2, \ldots, x^n\}\) be the \((n+1)\) dimensional vector space of polynomials of degree less than or equal to \(n\). Find the matrix representation for the linear operator \(T: P^n \to P^n\) defined by
   \[
   (Tp)(x) = \frac{dp}{dx}(x),
   \]
   in terms of the basis \(\{1, x, x^2, \ldots, x^n\}\).

3. Recall that a square matrix \(C\) is invertible if its rows or its columns are linearly independent. If \(\{\phi_1, \phi_2, \ldots, \phi_n\}\) and \(\{\psi_1, \psi_2, \ldots, \psi_n\}\) are two bases for the same vector space \(V\), and \(C\) is the \(n \times n\) matrix with coefficients \(c_{k,j}\) such that
   \[
   \phi_j = \sum_{k=1}^{n} c_{k,j} \psi_k, \quad j = 1, \ldots, n,
   \]
   prove that \(C\) is invertible.