Calculus

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First Lecture: What is calculus about and why should I study it?

The story of calculus began in the late 1600's with the revolutionary results of Isaac Newton (1642 - 1727) and Gottfried Leibniz (1646 - 1716) in understanding motion and rates of change. Some of the underlying concepts, however, such as the infinite and the infinitesimal, were thought about and articulated by the ancient Greeks, Zeno and Archimedes, well over 2000 years ago. The word *calculus* itself comes from Latin and means a small stone or pebble used in gaming, voting, or reckoning. It is hoped that after you have used this book, you will realize that this very long story is still being written, and is even more vibrant and essential today than it was over 300 years ago. With historical hindsight, it can be said that the development of calculus is certainly one of the greatest intellectual achievements of the past two millenia. Without calculus, most of the incredible advances in science and engineering which occurred in the twentieth century and have become part of everyday life, such as air and space travel, television, computers, weather prediction, medical imaging, nuclear bombs, wireless phones, the internet, microwave ovens, etc., could not have happened. Calculus provides the language and basic concepts used to formulate most of the fundamental laws and principles of the various disciplines throughout the physical, mathematical, biological, economic and social sciences, as well as electrical, mechanical, computer, bio-, civil, and materials engineering. Of course, within mathematics, calculus serves as the inescapable gateway to all higher level courses. Moreover, historically it is the seed which gave rise to many of the principal branches of mathematics, which themselves often have deep links to important areas of application throughout the sciences and engineering. Calculus is also an essential tool used widely thoughout business and industry, such as in the financial, insurance, transportation, manufacturing, and pharmaceutical industries, and in the development of computer, communications, and medical technologies. The role of mathematics, and calculus in particular, in serving as the "operating system" of science and engineering cannot be underestimated. It is for these reasons that students majoring in fields throughout the sciences, engineering, medicine, and business, as well as in mathematics, are required to take calculus. Clearly, the better you know the language of a country you may wish to visit, the more likely you'll be successful in meeting, communicating with, and learning from the people there. Similarly, the better you know the operating system of the field in which you intend to work, the better positioned you will be to succeed, be it in doing well in subsequent classes spoken in the language of calculus, or in formulating and solving problems in a job situation.

The main objects of study in calculus are functions, such as $f(x) = x^2$, $g(x) = \sin x$, and $k(x) = 2x$. One of the central questions which calculus addresses is to understand what we mean by the rate of change of a function f locally at a point x. In other words, how much does f change for small changes, or differences in x, and how can this be measured precisely? For functions whose graphs are straight lines, such as $k(x) = 2x$, the answer is provided by the slope of the line, which is 2 for $k(x)$, and is the same rate of change for all x. For example, if we start at $x = 1$ (or any x), and increase the value of x by $1/10$ to $x = 1.1$, then the value of k correspondingly increases by twice as much, or $1/5$. For functions whose graphs are not straight lines, such as $f(x) = x^2$ and $g(x) = \sin x$, the

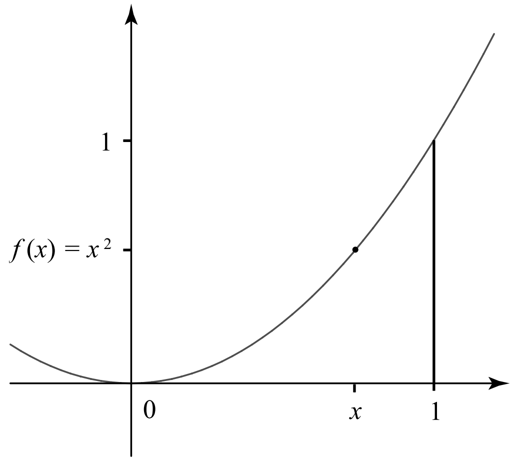

Figure 1: The graph of the function $f(x) = x^2$. The two central problems of calculus are finding the rate of change of a function at a point x, and finding the area under the graph of a function on an interval, such as [0,1] above.

rate of change (whatever this means exactly) depends on x. Clearly, the rate of change of $f(x) = x^2$ at $x = 1$ is much larger than at $x = 0$, where the function is nearly flat, as indicated in Fig. 1. A new function describing the rate of change of f, and how it depends on x, is so important that it is given its own special name and notation, and is called the derivative. The study of the derivative is often referred to as **differential calculus**. One way of denoting the derivative of $f(x)$ is by $f'(x)$, a function which is derived from $f(x)$, and whose value at x is the rate of change of f at x. Of course, for the linear function $k(x) = 2x$, then the rate of change function, or derivative, is given by $k'(x) = 2$, independent of x. The graph of $k'(x)$ is the horizontal line $y = 2$. The ideas used to find derivatives in general, and in particular for polynomial functions like $f(x) = x^2$, will be developed in Chapter 1. [You may want to examine the graphs of $f(x) = x^2$ and $g(x) = \sin x$ at a few values of x, estimate the rates of change at these points, and then try and figure out what the graphs of $f'(x)$ and $g'(x)$ look like.] Subsequent chapters will develop methods for finding derivatives of more complicated functions, and in more complicated situations, as well as a rigorous framework in which various properties of functions can be studied. Particular attention will be paid to exploiting information about the derivative to analyze and estimate the properties of the original function, and in exploring important applications of the derivative in science and engineering. Very often one encounters functions of more than one variable, for example $u(x, y) = x^2 - y^2$, and the development of the derivative in such cases leads to the investigation of new phenomena and ideas, which will be addressed in later chapters of the book.

The other central question which calculus addresses is how to find the total area under the graph of a function on an interval [a,b], such as the area under $f(x) = x^2$ on [0,1] as in Fig. 1. This type of global measure of a function is called an integral. For the linear function $k(x) = 2x$ on [0,1], we already know how to find the area under the graph, since this region is just a triangle of known dimensions. [What is its area?] However, the regions under functions such as $f(x) = x^2$ on [0,1] or $g(x) = \sin x$ on $[0, \pi]$ are not simple shapes whose areas we already know how to find, such as rectangles, triangles and circles. [Can you think of a way of estimating the area under $f(x) = x^2$ on [0,1], perhaps using shapes whose area you already know?] The answers to such questions are developed in **integral**

calculus. It is fascinating that the global question about the area under $f(x)$ on an interval is intimately connected to the seemingly unrelated question about the local rate of change of $f(x)$ described above. This deep, and perhaps unexpected, connection between the two great problems of calculus is one of its most important results, called the fundamental theorem of calculus. Generalizing the notion of the integral, as well as the fundamental theorem, to more than one variable, again leads to many new and important ideas used throughout higher mathematics, the sciences, and engineering.

As you can see, calculus provides us with a framework to analyze the most basic and essential properties of functions. The reason calculus has such a monopoly in describing our world is that almost any quantitative model of a physical, chemical, biological, engineering, industrial, or financial system involves the use of functions. Moreover, almost any type of analysis of these functions used for understanding or predicting the behavior of the system, particularly how it evolves in time, will invariably involve calculus. This is not surprising, however, given the basic and fundamental nature of the tools developed in calculus. For example, in physics and engineering one regularly encounters functions describing the position of a particle or object as a function of time t, such as $x(t)$ for motion in one dimension. The function $x(t)$ might represent the height of a falling rock, or the position of a mass hanging by a spring, which is stretched and then let go. As we will see very soon, the derivative $x'(t)$ is what we normally refer to as the *velocity* $v(t) = x'(t)$ of the object, or the rate of change of the position. Particular attention will be paid in this book to motion examples, not only due to their fundamental importance, but because most everyone has a lot of experience with motion and velocity in everyday life, and hopefully a good intuition about it. In fact, in this context it is natural to consider the rate of change, or derivative $v'(t)$ of the velocity $v(t)$, which we know as the *acceleration* $a(t) = v'(t)$, from common experience in cars, airplanes, and amusement park rides (such as the thrill of going from 0 to 60 mph in less than 5 seconds in a sports car, or of being steam catapulted off the deck of an aircraft carrier). In terms of the original function $x(t)$, the derivative is taken twice, so that the acceleration $a(t) = v'(t) = x''(t)$ is the second derivative of the position $x(t)$.

Other examples of functions encountered in science and engineering include the current $I(t)$ in an electrical circuit, perhaps being used in a computer or in a radar or MRI system, the average temperature $T(z)$ in a region of the ocean as a function of depth z, the size $N(t)$ of a rapidly growing viral population, the electric field strength $E(t)$ received by a wireless phone during a call, or by the cornea during laser eye surgery, the sound pressure $P(t)$ received by your ear during a favorite song, the fluid velocity $v(x, y)$ in a cross-section of a pipe, artery, or tornado, or on the surface of an airplane wing or submarine hull, the gene activity function A(g) along a DNA strand, measuring how many times a gene g is transcribed, or called, during a cellular process, a company's stock price $S(t)$ or total revenues $R(t)$, the density of water $\rho(T)$ as a function of its temperature T, particularly near where water boils at $100°C$, the surface air pressure $p(x, y, t)$ over the United States as a function of time or the thickness $H(x, y, t)$ of the Antarctic ice sheet with time, the concentration $C(t)$ of salt during a chemical reaction, the electrical action potential $V(t)$ of a neuron, the amount of strain $e(x, y, z)$ produced in a material under stress, and the wave function $\Psi(x, y, z)$ describing the quantum state of an electron in an atom.

As mentioned above, calculus provides the language and concepts used to formulate most of the fundamental laws and principles of science and engineering. This is because these laws and principles are usually, and most naturally, expressed locally in terms of rates of change or dervatives, or globally in terms of integrals (or in some cases in terms of derivatives *and* integrals). If a law is expressed locally, which is most common, then it typically has the form of an equation involving the derivatives of a function, as well as perhaps the function itself. These equations are called **differential equations**. For example, finding a function $y(t)$ that satisfies the differential equation $y'(t) = y(t)$, or finding a function $x(t)$ that satisfies $x''(t) + 4x(t) = 0$, are interesting and important problems. Just as with calculus itself, it is difficult to underestimate the importance of differential equations. They play the central role in expressing the fundamental principles and laws governing our world, as well as in understanding and predicting the behavior of much of what is in it, be it naturally occurring or man-made. A principal reason for the importance of differential equations in the description of our world is that Newton's famous law, $F = ma$ or *Force* = *mass* × *acceleration*, which underlies so much of our understanding of motion and dynamics, is a differential equation. The acceleration $a(t) = v'(t) = x''(t)$ is the second derivative of the position $x(t)$. Such differential equations involving the second derivative (but no higher derivatives), and perhaps the first derivative and the function itself, are called *second order* differential equations, like $x''(t) + 4x(t) = 0$, or $x''(t) = -32$. [Can you solve either of these equations?] Of course, equations involving only the first derivative and the function itself, like $y'(t) = y(t)$, are called *first order* differential equations. Because so much of the description of our world rests on Newton's laws, as well as on other laws related to or depending in some way on Newton's framework, second order differential equations occupy a special place throughout mathematics, the sciences, and engineering. The pervasiveness of this second order structure has dictated to some extent which types of differential equations have received the most attention from theoretical, computational and experimental researchers over the past couple hundred years.

Another striking example of how mathematics, and multivariable calculus and differential equations in particular, facilitates the quantitative description of our world is Maxwell's equations. In 1865 James Clerk Maxwell saw a fundamental inconsistency in a set of differential equations describing the basic phenomena of electromagnetism. This set of second order differential equations expressed mathematically the known experimental facts at the time. By adding a "missing" term to the equations and transforming them into a consistent set, he was able to make the startling prediction that light is an electromagnetic wave, and that electromagnetic waves of all frequencies could be produced. This discovery spurred a tremendous amount of experimental and theoretical research on electromagnetism in the late 1800's, and laid the groundwork for much of what we take for granted in the 21st century. Indeed, Maxwell's equations provide the framework in which all electromagnetic wave phenomena are analyzed, such as the propagation, reflection and scattering of light, radar, microwaves, x-rays, and lasers, and they govern the propagation of electrical nerve impulses and electrical activity in the heart and brain as well. It is amazing that this beautiful set of equations can be viewed as a statement, appropriate to the electromagnetic field, of the fundamental theorem of calculus and related results about functions of several variables. Further generalizations and applications of these ideas to studying the geometry of higher dimensional "surfaces," called *manifolds*, eventually came to scientific fruition when they served as the mathematical foundation of Einstein's theories of relativity.

These startling developments in the understanding of our world, and in creating the mathematical framework for analyzing it, laid the groundwork for more surprising advances. Ernest Rutheford used basic ideas in calculus to investigate the internal structure of the atom, which culminated in the discovery of the atomic nucleus. He was awarded the Nobel Prize for Chemistry in 1908 for this work. Further investigations of atomic phenomena revealed that in many experiments, matter, such as electrons, exhibit wave-like behavior, which led to the development of the theory of *quantum mechanics*. Since it's generally more difficult to specifically localize a wave, such a realization shifted the point of view away from solving Newton's deterministic differential equations for *the* particle position $x(t)$, often referred to as *classical mechanics*, at least for atomic scale phenomena. A new function $\Psi(x,t)$, called the wave function, became the focus of interest. For an electron in one dimension, it describes, among other things, the probability of finding the electron in an interval [a,b] at time t. In fact, this probability is found with an integral. In 1926 Erwin Schrödinger developed his famous wave equation for $\Psi(x,t)$, for which he later won the Nobel Prize for Physics in 1933. This second order differenential equation has connections to Newton's law, yet is also similar to the classic equation of heat conduction, except it involves the number $i = \sqrt{-1}$, which stands for imaginary (not *impossible*!). Its solutions can then exhibit wave behavior. The development of quantum theory laid the foundation for the discovery of DNA, the transistor, semiconductors (the materials used to make the brains of current computers and mobile phones), nuclear weapons, MRI imaging, and radiation treatment for cancer.

There are myriad other examples where mathematics, and calculus and differential equations in particular, has enabled a major scientific or technical advance. One of the interesting ways in which this happens is through the *transference* of ideas and techniques that are developed in one field of study, and are then subsequently applied to another. The reason this is possible is because the language and methods of mathematics, and calculus in particular, are **universal**. If they work on one function of a certain type, or which satisfies a certain type of equation, then similar methods will usually work on another function of that type, even if it comes up in a completely different field. For example, mathematical techniques used to study air flow over an airplane or insect wing can be used to study the flow of blood in the heart, as the underlying second order differential equations are the same. In chemistry one often needs to solve Schrödinger's equation for the wave function describing systems with many interacting particles, as well as with only a few. The ideas behind techniques developed to exactly calculate, or more realistically, approximate the solutions are often related to techniques developed to solve similar problems throughout the sciences. Such problems include studying the propagation of light, microwaves, or sound through media with many interacting scatterers, like the atmosphere, ocean, or the human body, and the behavior of Brownian motion and diffusion processes, like the motion of a pollen grain on a droplet of water. If a mathematical theory, based heavily on calculus, was developed originally to understand the flow of electrons through semiconductors, the basic building blocks of silicon based computers, yet happens to also shed light on almost the same functions describing fluid flow through sea ice arising in the study of bacteria and algae living inside the salty, liquid inclusions lacing the ice which covers the polar oceans of earth, so much the better.

Similarly, *harmonic*, *spectral*, or *Fourier analysis*, a branch of mathematics which arose out of calculus originally in connection with solving the second order differential equation for heat conduction, perhaps gained most prominence in the late 1800's by providing a mathematical foundation for the frequency or spectral analysis of light, sound and music. Fourier analysis and related areas of mathematics now lie at the heart of many types of medical imaging, digital sound and video recordings, oil and mineral recovery, climate analysis, electrical brain wave analysis, image compression for transmission across the internet, design of components in telecommunications networks, remote sensing of the earth from satellites, as well as a broad range of techniques used in scientific computing and algorithm design. It is interesting to note that the heat equation has also provided the foundation for a theory of pricing *options* on a stock, which are called "derivative securities" or "financial derivatives," since their prices depend on the underlying stock price and how it fluctuates, sometimes seemingly randomly over small time scales, as well as perhaps makes larger moves over longer time scales. Fischer Black and Myron Scholes found through mathematical analyis that option prices typically obey a type of heat or diffusion equation, now called the Black-Scholes model, for which the Nobel Prize in Economics was awarded in 1997.

The operating system for analyzing functions provided by calculus is what allows a particular set of ideas such as Fourier analysis and the heat equation to achieve such wide influence. From experience with personal computers you know that any application which works on one computer running Windows (or MAC OS or Linux) as its operating system, will (in principle) work on any other computer running the same operating system, even if that computer is on the other side of the world, down the hall, or on a trip to Jupiter. It is this type of standardization, as well as standardization of protocols for how computers communicate with one another, which led to the explosion of personal computers and the internet in the late 1900's. Similarly, calculus standardized how functions in *any* field are quantitatively analyzed, in terms of how to calculate rates of change and integrals. Calculus provided a universal platform upon which fundamental laws can be formulated and analyzed, as well as a language through which scientists in different fields studying similar types of functions could communicate and transfer methods of analysis and solution. The development of calculus gave rise to an explosion of inquiry and findings throughout the sciences and engineering. These advances most certainly enabled the Industrial Revolution of the early 1900's, as calculus finally provided a systematic way to study the dynamic and not just the static, the dawn of the Nuclear Age in the 1940's, as well as the rise of the Information Age in the late 1900's.

As a vivid illustration of the impact of these advances, suppose that Newton (or his estate) could have received licensing fees or royalties from the sale of all the products and services whose creation used the derivative, the integral, or his related laws of motion in any way. After all, Microsoft receives fees for the use of *its* operating system, Windows. For example, as we have pointed out, all electromagnetic wave phenomena are analyzed and exploited commercially within the framework of Maxwell's *differential* equations, which rest upon Newton's framework. The entire telecommunications and internet industries owe their very existence to the development of calculus, as well as many subsequent breakthroughs such as Maxwell's equations. More generally, Newton effectively provided us with a platform upon which the bulk of the technical infrastructure of our modern world has been built. Newton's fortune would then dwarf that of Bill Gates, co-founder of Microsoft, many times over, even if Newton's cut were 0.1%. Indeed, part of Microsoft's revenues would have to be paid in fees to Newton's estate!

From the above discussions, you now have some idea what calculus is about, and why it's very important to study and understand it well. Hopefully you have glimpsed a bit of the depth and *gravitas* of calculus, although at this point you are only seeing the tip of this iceberg. As we progress, I will try to show you what lies beneath the topics we're studying, how they have laid the groundwork for major advances throughout science and engineering, and where they might lead to in higher mathematics. Hopefully you will come to realize that the light of calculus penetrates even into the most remote reaches of our world, illuminating understanding, and providing a framework in which one may ask, and *answer*, quantitative questions about how and why things work the way they do.

Per ardua ad astra.