Brownian motion & thermal capacity

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(joint work with Yimin Xiao)

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Brownian motion & fractals

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  - Much more (1970+)

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- Equivalently, $^\forall$ compactly-supported probab. meas. μ on $E \times F$,

$$\iint \frac{e^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{d/2}}\,\mu(\mathrm{d}s\,\mathrm{d}x)\,\mu(\mathrm{d}t\,\mathrm{d}y) = \infty.$$

(K–Xiao, 2011; Watson, 1974, 1977)

Hausdorff dimension in space-time

- Let $\Sigma := \mathbf{R}_+ \times \mathbf{R}^d$ [space-time]., metrized by the parabolic metric

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  - If $\dim_{\mathrm{H}}(E \times F\,;\varrho) > d$, then $W(E) \cap F \neq \varnothing$ w.p.p.

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  - If $\dim_{\mathrm{H}}(E \times F\,;\varrho) < d$, the $W(E) \cap F = \varnothing$ a.s.

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The easier case ($d \geq 2$)

Theorem

If $d \geq 2$, *then* $\left\| \dim_{\mathrm{H}}(W(E) \cap F) \right\|_{L^\infty(\mathrm{P})} = \left(\dim_{\mathrm{H}}(E \times F\,; \varrho) - d \right)_+ .$

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If $d \geq 2$, then $\|\dim_{\rm H}(W(E) \cap F)\|_{L^\infty(\mathrm{P})} = (\dim_{\rm H}(E \times F\,;\varrho) - d)_+$.

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- Theorem false for $d = 1$; e.g., $E := [0\,,1]$, $F := \{0\}$.

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- $\dim_{\rm H}(E \times F\,;\varrho) - d$ is the slack in the Taylor–Watson condition (codimension?)
- Theorem false for $d = 1$; e.g., $E := [0\,,1]$, $F := \{0\}$.
- Then, $\dim_{\rm H}(W(E) \cap F) = 0$, $\dim_{\rm H}(E \times F\,;\varrho) - 1 = 1$.

The harder case ($d = 1$)

Theorem

$$\left\|\dim_{\mathrm{H}}(W(E)\cap F)\right\|_{L^\infty(\mathrm{P})} = \sup\left\{\beta > 0 : \inf_{\mu\in M_1(E\times F)} \mathcal{E}_\beta(\mu) < \infty\right\},$$

where

$$\mathcal{E}_\beta(\mu) := \iint \frac{\mathrm{e}^{-|x-y|^2/(2|t-s|)}}{|t-s|^{1/2}\cdot|x-y|^\beta}\,\mu(\mathrm{d}s\,\mathrm{d}x)\,\mu(\mathrm{d}t\,\mathrm{d}y).$$

Proof when $d \geq 2$

$\|\dim_{\mathrm{H}}(W(E)\cap F)\|_{L^\infty(\mathrm{P})} = (\dim_{\mathrm{H}}(E \times F\,;\varrho) - d)_+$

- $W(E) \cap F = W(E \cap W^{-1}(F))$.

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- Therefore, $\dim_{\mathrm{H}}(W(E) \cap F) = 2\dim_{\mathrm{H}}(E \cap W^{-1}(F))$.

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- Let $X_\alpha :=$ an independent symmetric α-stable Lévy process with $\alpha \in (0\,,1)$. A bound:

$$\mathrm{P}\left\{ X_\alpha[0\,,1] \cap \overbrace{[t - r^2\,, t + r^2]}^{I} \neq \varnothing \right\} \leq \text{const} \cdot r^{2(1-\alpha)}.$$

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- A 2nd bound: $\mathrm{P}\left\{ W(I) \cap \overbrace{B(x\,,r)}^{J} \neq \varnothing \right\} \leq \text{const} \cdot r^d$.

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- A 2nd bound: $\mathrm{P}\left\{ W(I) \cap \overbrace{B(x\,,r)}^{J} \neq \varnothing \right\} \leq \text{const} \cdot r^d$.
- $\therefore \mathrm{P}\left\{ I \cap W^{-1}(J) \cap X_\alpha[0\,,1] \neq \varnothing \right\} \leq \text{const} \cdot r^{d+2(1-\alpha)}$.

Proof when $d \geq 2$

$\| \dim_{\mathrm{H}}(W(E) \cap F) \|_{L^\infty(\mathrm{P})} = (\dim_{\mathrm{H}}(E \times F\,; \varrho) - d)_+$

- Cover $E \times F$ with "parabolic balls" $\{E_j \times F_j\}_{j=1}^{\infty}$ of the form $[t - r^2\,, t + r^2] \times B(x\,, r)$.

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- $\therefore\ \mathrm{P}\{E \cap W^{-1}(F) \cap X_\alpha[0\,,1] \neq \varnothing\} \leq C\mathrm{H}_{d+2(1-\alpha)}(E \times F\,; \varrho)$. (Vitali covering)

Proof when $d \geq 2$

$\| \dim_{\mathrm{H}}(W(E) \cap F) \|_{L^\infty(\mathrm{P})} = (\dim_{\mathrm{H}}(E \times F\,; \varrho) - d)_+$

- Therefore, $\dim_{\mathrm{H}}(E \times F; \varrho) < d + 2(1 - \alpha)$ implies

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- Therefore,

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- Therefore, $\dim_{\mathrm{H}}(E \times F; \varrho) < d + 2(1-\alpha)$ implies

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- Kaufman's theorem:

$$\dim_{\mathrm{H}}(W(E) \cap F) \leq \left(\dim_{\mathrm{H}}(E \times F\,; \varrho) - d\right)_+ .$$

Proof when $d \geq 2$

$\dim_{\mathrm{H}}(W(E) \cap F) = \dim_{\mathrm{H}}(E \times F\,; \varrho) - d$

- **Theorem:** $E \cap W^{-1}(F) \cap X_\alpha[0\,,1] \neq \varnothing$ with pos. probab. if

$$\inf_{\mu \in M_1(E \times F)} \iint \underbrace{\frac{e^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{(d/2)+(1-\alpha)}}}_{\mathcal{J}(d+2(1-\alpha))} \mu(\mathrm{d}s\,\mathrm{d}y)\,\mu(\mathrm{d}t\,\mathrm{d}x) < \infty.$$

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- If $\dim_{\mathrm{H}}(E \times F\,; \varrho) > d + 2(1-\alpha)$ then $E \cap W^{-1}(F) \cap X_\alpha[0\,,1] \neq \varnothing$ w.p.p.

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- $\mathcal{J}(\beta) \leq \text{const} \cdot \varrho((s\,,y)\,;(t\,,x))^{-\beta}$.
- $\iint \mathcal{J}(d+2(1-\alpha))\,d(\mu\times\mu) \leq \text{const} \times$ Riesz energy of μ in dimension $d+2(1-\alpha)$.
- If $\dim_{\rm H}(E \times F\,;\varrho) > d + 2(1-\alpha)$ then $E \cap W^{-1}(F) \cap X_\alpha[0\,,1] \neq \varnothing$ w.p.p.
- McKean's theorem (1955):
$\|\dim_{\rm H}(E \cap W^{-1}(F))\|_{L^\infty(\mathrm{P})} \geq \frac12\{\dim_{\rm H}(E \times F\,;\varrho) - d\}_+$.

An aside

- Actually showed: $^\forall d \geq 1$,

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- Hawkes conjectured a formula when $W \leftrightarrow \mathrm{SS}(\alpha)$. Conjecture is correct.

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- Theorem of Hirsch–Song (1994):

$$\mathrm{P}\left\{\mathfrak{X}_\alpha(\mathbf{R}^N_+) \cap G \neq \varnothing\right\} > 0 \quad \text{iff} \quad \mathrm{Cap}_{d-\alpha N}(G) > 0.$$

Comments on the case $d = 1$

- Let $X_1, X_2, \ldots, X_N$ be N i.i.d. α-stable processes in $\mathbf{R}^d$
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$$\dim_{\mathrm{H}} G > d - \alpha N \quad \Rightarrow \quad \mathfrak{X}_\alpha(\mathbf{R}^N_+) \cap G \neq \varnothing \quad \text{w.p.p.}$$
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- Computes $\dim_{\mathrm{H}} G$ (Taylor, 1961; K, 2001; K–Xiao–Shieh, 2008)

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- **Theorem:** If $d > \alpha N$ then $W(E) \cap F \cap \mathcal{X}_\alpha(\mathbf{R}_+^N) \neq \varnothing$ w.p.p. iff

$$\exists \mu \in M_1(E \times F) : \quad \mathcal{E}_{d-\alpha N}(\mu) < \infty,$$

where

$$\mathcal{E}_\beta(\mu) := \iint \frac{\mathrm{e}^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{d/2}\|y-x\|^\beta}\,\mu(\mathrm{d}s\,\mathrm{d}y)\,\mu(\mathrm{d}s\,\mathrm{d}x).$$

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- **Open problem:** "Why" is it that when $d \geq 2$,

$$\sup\left\{\beta > 0 : \inf_{\mu \in M_1(E \times F)} \mathcal{E}_\beta(\mu) < \infty\right\} = \{\dim_{\mathrm{H}}(E \times F\,; \varrho) - d\}_+ ?$$