Stirling's Formula and Laplace's Method
OR
How to Put Your Calculus to Good Use

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Jumbles (aka Permutations)

- There are 2 jumbles of "OF" ("OF" and "FO");

- There are 6 jumbles of "OFT" ("OFT", "FTO", "FOT", "OTF", "TOF", and "TFO");

- There are 24 jumbles of "SOFT" (!)

  ⋮

In general, there are $N! = N(N-1)\cdots 1$ jumbles of N distinct objects. This is read as "N factorial".

$N! = N \times (N-1) \times \cdots \times 1$ is a sequence that grows large rapidly as N grows, viz.,

$$
\begin{aligned}
1! &= 1 \\
2! &= 2 \\
3! &= 6 \\
4! &= 24 \\
5! &= 120 \\
&\vdots \\
20! &= 2.432902008 \times 10^{18} \\
40! &= 8.16 \times 10^{47} \\
100! &= 9.33 \times 10^{157} \\
400! &= \text{Error}
\end{aligned}
$$

$$\mathfrak{S}(N) = \sqrt{2\pi}e^{-N}N^{N+\frac{1}{2}}.$$

$$\begin{aligned}
\mathfrak{S}(20) &= 2.422786847 \times 10^{18} \\
\mathfrak{S}(40) &= 8.14217264483 \times 10^{47} \\
\mathfrak{S}(100) &= 9.32 \times 10^{157} \\
\mathfrak{S}(400) &= \text{Error}
\end{aligned}$$

Stirling's Formula (De Moivre 1730)

$$\lim_{N\to\infty} \frac{N!}{\sqrt{2\pi}e^{-N}N^{N+\frac{1}{2}}} = 1.$$

In simple words, for large N,

$$N! \simeq \sqrt{2\pi}e^{-N}N^{N+\frac{1}{2}}.$$

A Little History

Stirling's formula was found by Abraham de Moivre and published in *"Miscellenea Analytica"* 1730. It was later refined, but published in the same year, by J. Stirling in *"Methodus Differentialis"* along with other little gems of thought. For instance, therein, Stirling computes the area under the *Bell Curve*:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi},$$

$\cdots$ we will come back to this computation of James Stirling.

The Gamma Function Representation

$$N! = \int_0^\infty e^{-t} t^N \, dt.$$

To see this, let $u = t^N$ and $dv = e^{-t} \, dt$, and integrate by parts:

$$\begin{aligned} \int_0^\infty e^{-t} t^N \, dt &= -t^N e^{-t} \Big|_0^\infty + N \int_0^\infty e^{-t} t^{N-1} \, dt \\ &= N \int_0^\infty e^{-t} t^{N-1} \, dt. \end{aligned}$$

So, we can iterate this in N:

$$\begin{aligned} \int_0^\infty e^{-t} t^N \, dt &= N(N-1) \int_0^\infty e^{-t} t^{N-2} \, dt \\ &= N(N-1)(N-2) \int_0^\infty e^{-t} t^{N-3} \, dt \\ &\vdots \\ &= N! \int_0^\infty e^{-t} \, dt \\ &= N!. \end{aligned}$$

The Gamma Function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \qquad x > 1.$$

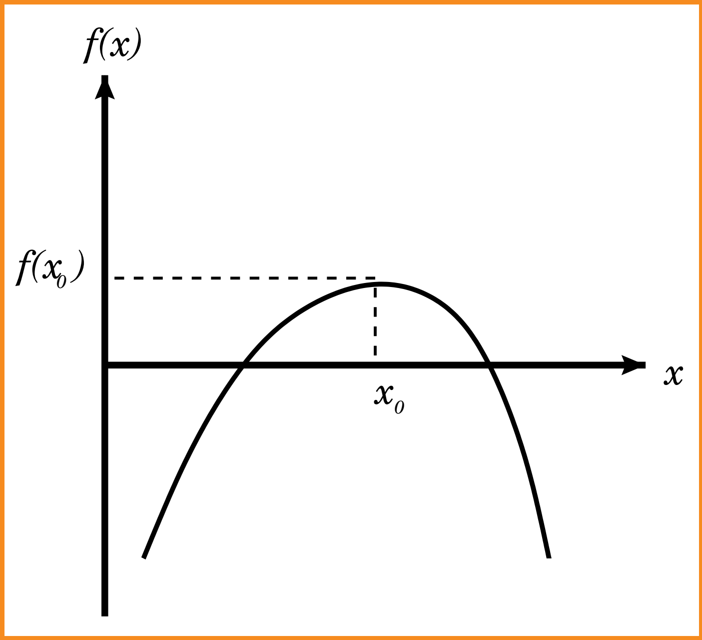
Check, by changing variables, that

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-x^2/2}\, dx.$$

Stirling showed that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Thus, this is equivalent to the area under the bell curve $= \sqrt{2\pi}$ (check!)

Laplace's Method

Goal: estimate $\int_{-\infty}^{\infty} e^{Nf(x)}\, dx$ for large N, where f looks like



It can be shown that for such a function, all of the main contribution to the integral comes from the x values near x_0, say, $(1-\varepsilon)x_0 \le x \le (1+\varepsilon)x_0$. The contribution from the other x values is asymptotically (i.e., as $N \to \infty$) is negligible. Here, ε is an arbitrary positive constant.

So, for f of the form shown in the previous figure, and for large N,

$$\int_{-\infty}^{\infty} e^{Nf(x)}\,dx \simeq \int_{(1-\varepsilon)x_0}^{(1+\varepsilon)x_0} e^{Nf(x)}\,dx. \qquad (1)$$

Pick ε small and use Taylor's expansion near x_0:

$$f(x) \simeq f(x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2,$$

and recall that $f''(x_0) < 0$ (Max. at x_0). Plug this in Eq. (1), to obtain

$$\int_{-\infty}^{\infty} e^{Nf(x)}\,dx \simeq e^{Nf(x_0)} \int_{(1-\varepsilon)x_0}^{(1+\varepsilon)x_0} e^{\frac{N}{2} f''(x_0)(x-x_0)^2}\,dx.$$

To re-iterate

$$\int_{-\infty}^{\infty} e^{Nf(x)}\,dx \simeq e^{Nf(x_0)} \int_{(1-\varepsilon)x_0}^{(1+\varepsilon)x_0} e^{\frac{N}{2}f''(x_0)(x-x_0)^2}\,dx.$$

Recall that $f''(x_0) < 0$, and change variables $(y = \sqrt{-Nf''(x_0)}(x - x_0))$:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{Nf(x)}\,dx &\simeq e^{Nf(x_0)} \int_{-\varepsilon\sqrt{-Nf''(x_0)}}^{+\varepsilon\sqrt{-Nf''(x_0)}} e^{-\frac{1}{2}y^2}\,dy \times \\ &\quad \times \frac{1}{\sqrt{-Nf''(x_0)}} \\ &\simeq e^{Nf(x_0)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy \times \\ &\quad \times \frac{1}{\sqrt{-Nf''(x_0)}} \\ &= \frac{\sqrt{2\pi}e^{Nf(x_0)}}{\sqrt{-Nf''(x_0)}}.\end{aligned}$$

Back to Stirling

Since $N! = \int_0^\infty e^{-t} t^N \, dt$ from the first slide,

$$N! = \int_0^\infty e^{-t + N \ln t} \, dt.$$

Change variables ($s = Nt$) to get

$$\begin{aligned} N! &= N^{N+1} \int_0^\infty e^{-N(\ln s - s)} \, ds \\ &= N^{N+1} \int_0^\infty e^{N f(s)} \, ds, \end{aligned}$$

where $f(s) = \ln(s) - s$. Check that f is of the desired form, $x_0 = 1$, $f(x_0) = -1$, $f''(x_0) = -1$, and now we derive the Striling's formula from Laplace's method easily:

$$N! \simeq N^{N+1} \frac{\sqrt{2\pi} e^{N f(x_0)}}{\sqrt{-N f''(x_0)}} = N^{N+1} \frac{\sqrt{2\pi} e^{-N}}{N^{\frac{1}{2}}},$$

which equals $\sqrt{2\pi} e^{-N} N^{N+\frac{1}{2}}$.

Easy Exercise

Check, by using Laplace's method, that

$$\int_0^\pi x^N \sin(x)\, dx \simeq \pi^{N+2} N^{-2}.$$

Harder Exercise

(Essentially the local de Moivre–Laplace central limit theorem)

Check, by using Laplace's method, that for even numbers N,

$$\frac{N!}{(N/2)!} \simeq 2^N \sqrt{\frac{2}{\pi N}},$$

where, in both exercises, $f(N) \simeq g(N)$ means that $f(N)/g(N) \to 1$, as $N \to \infty$.