Estimates for the Laws of Functionals of a Brownian Sheet:

Recent Progress and Open Problems

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Local Structure of the Sheet

- Brownian sheet Let $B_t = (B_t^1, \ldots, B_t^d)$, where B^i's are i.i.d. centered, cont. Gauss. with

$$\mathbb{E}\{B^1_{s,t} \cdot B^1_{u,v}\} = \min(s,u) \times \min(t,v).$$

- Local theory (Dalang and Walsh '90–'93; ; Ehm '83; Orey and Pruitt '74): fix s, t

$$B_{s+\varepsilon_1, t+\varepsilon_2} = B_{s,t} + t^{\frac{1}{2}}\beta_1(\varepsilon_1) + s^{\frac{1}{2}}\beta_2(\varepsilon_2) + \mathfrak{B}_{\varepsilon_1,\varepsilon_2},$$

where $(\beta_1, \beta_2, \mathfrak{B})$ are independent processes, all independent of $B_{s,t}$; β_i's are Brownian motions, and $\mathfrak{B}$ is a Brownian sheet.

$\therefore$ if $\varepsilon_1, \varepsilon_2$ are small,

$$B_{1+\varepsilon_1, 1+\varepsilon_2} \approx B_{1,1} + \overbrace{\beta_1(\varepsilon_1) + \beta_2(\varepsilon_2)}^{\text{additive BM}},$$

There are also analytical consequences, since additive BM "corresponds" to

- the operator $(\frac{1}{2}\Delta)^2$ (Kh-Shi '00; Kh-Xiao '01);
- the 2-par convolution semigroup $(s, t) \mapsto H_s H_t$, where H denote heat semigroup; (*ibid.*);
- the operator $\frac{1}{2}\Delta - u^2$ (*ibid.* + Le Gall '00).

Here, we are interested in non-local results and wish to show-case the viewpoint that B can also be thought of as a stream of interacting Brownian motions, where the nature of the interaction seems to be problem-specific. Often, this can be translated to problems about a single Brownian motion in *really high* dimensions.

Open Problem 0* *Make one non-trivial calculation, where nontrivial means anything that uses deeper things than the linear structure of the Gauss space. E.g., two examples from math. statistics:*

- *find the law of* $\sup_{[0,1]^2} B$ *(circa '40's-'50's)*
- *find the law of* $\iint_{[0,1]^2} \mathbf{1}_{(B_{s,t}>0)}\, ds\, dt$ *(circa late '60's or early '70's)*

Relation to Eigenvalue Problems

Theorem 1 **(Csáki–Kh–Shi '00; Li–Shao '01)** *There are c_1 and c_2 such that for all $\varepsilon \in (0,1)$,*

$$e^{-c_1|\ln \varepsilon|^2} \leq \mathbb{P}\{\sup_{[0,1]^2} B < \varepsilon\} \leq e^{-c_2|\ln \varepsilon|^2}.$$

Note that for a Brownian motion β,

$$\ln \mathbb{P}\{\sup_{[0,1]} \beta < \varepsilon\} \sim \ln \varepsilon.$$

We intend to argue that, in the present context, the supremum of Brownian sheet looks roughly the same as the supremum of Brownian motion in dimension $d \approx 2 \log(1/\varepsilon)$. In fact, Theorem 1 can be related to an eigenvalue estimate for a cone in $d \approx 2 \log(1/\varepsilon)$.

Not much happens away from the axes, as we shall heuristically argue in the next slide. Take this for granted to see that heuristically,

$$\sup_{[0,1]^2} B < \varepsilon \quad \text{``}\Longleftrightarrow\text{''} \quad \sup_{j\geq 0}\ \sup_{0\leq v\leq 1} B_{e^{-j},v} \leq \varepsilon.$$

Thus, writing β for Brownian motion,

$$\begin{aligned}\mathbb{P}\{\sup_{[0,1]^2} B < \varepsilon\} &\approx \prod_{j=0}^{\infty} \mathbb{P}\{\sup_{v\in[0,1]} B_{e^{-j},v} \leq \varepsilon\} \\ &= \prod_{j=0}^{\infty} \mathbb{P}\{\sup_{[0,1]} e^{-\frac{j}{2}}\beta \leq \varepsilon\} \\ &\approx \prod_{j=0}^{\infty} [\varepsilon e^{\frac{j}{2}} \wedge 1],\end{aligned}$$

where $\approx$ does not mean anything precise.

Open Problem 1* *Can this argument be made rigorous? Is there a limiting constant? Is it a* $\frac{1}{2}$*?*

The following proves, anectodally, that not much happens off of the axes.

Theorem 2 (Csáki–Kh–Shi '01) *For all* $0 < a < b$, *there exists a* $\theta(a,b) = \theta > 1$, *such that*

$$\ln \mathbb{P}\{\sup_{[a,b]\times[0,1]} B < \varepsilon\} \sim \theta \ln \varepsilon, \quad \textit{as } \varepsilon \to 0.$$

Thus, the cdf all the way upto the axis, dies like $e^{-|\log \varepsilon|^2}$—faster than any power law—while away from an axis, there is a power law.

Open Problem 2* *Does* $\theta(a,b) \to \infty$ *as* $a \to \infty$? *If so, how fast?* $\cdots$ *Eigenvalue estimates in high dimensions.*

Towards a Feynman–Kac Formula

Define the $\mathfrak{S}$ojourn measure,

$$\mathfrak{S}(F) = \iint_{[0,1]^2} \mathbf{1}_{(B_{s,t} \in F)}\, ds\, dt.$$

Theorem 3 (Kh–Pemantle '01) *There are c_1 and c_2 such that for all $\varepsilon \in (0,1)$,*

$$e^{-c_1 |\ln \varepsilon|^2} \leq \mathbb{P}\{\mathfrak{S}(0,\infty) < \varepsilon\} \leq e^{-c_2 |\ln \varepsilon|^2}.$$

For a Brownian motion β,

$$\ln \mathbb{P}\{\mathfrak{S}_\beta(0,\infty) < \varepsilon\} \sim -\ln \varepsilon.$$

This is from the arc-sine law. In particular, there is no arc-sine law for the sheet. This was posed by R. Pyke.

Open Problem 3* *Is there a limiting constant?*

For most interesting F with $F^\circ \ni 0$, the Feynman–Kac tells us that $\mathfrak{S}_\beta(F)$ (for Brownian motion) has exponential tails. For Brownian sheet, the story is more complicated still, e.g.,

Theorem 4 (Kh–Pemantle '01) *There are c_1 and c_2 such that for all $\varepsilon \in (0,1)$,*

$$e^{-c_1 \frac{|\ln \varepsilon|^2}{\varepsilon}} \leq \mathbb{P}\{\mathfrak{S}(-1,1) < \varepsilon\} \leq e^{-c_2 \frac{|\ln \varepsilon|}{\varepsilon}}.$$

Open Problem 4* *What is the sharp rate? Is there a limiting constant? Is there a finite-dimensional "Feynman–Kac" type formula?*

Explosion of Local Times Along Lines

Let W_u denote the $C[0,\infty)$-valued Brownian motion

$$W_u(v) = B_{u,v}, \quad u, v \geq 0.$$

The process W_u has local times $L_t^0(W_u)$ for each u, viz.,

$$L_t^0(W_u) = \int_0^t \delta_0(W_u(v))\, dv.$$

This is the semi-mart. local time and

$$[W_u]_v = uv.$$

So, $L_1^0(W_u)$ has the same law as $u^{-\frac{1}{2}} L_1^0(W_1)$, i.e., Brownian local time times $u^{-\frac{1}{2}}$. Thus,

$$\lim_{u \to 0} L_1^0(W_u) = +\infty, \qquad \text{in probability.}$$

When counting excursions of the sheet in '95, I asked if the above holds almost surely. Now, by scaling, if $u_k \to 0$ rapidly enough, e.g., if $u_k = e^{-k}$, using scaling and Borel-Cantelli:

$$\lim_{k\to\infty} L_0^1(W_{u_k}) = +\infty, \quad \text{almost surely.}$$

Thus, a.s. convergence $\Leftrightarrow$ nothing much happens to the oscillations of $u \mapsto L_1^0(W_u)$ in the block $[u_{k+1}, u_k]$. This rings of a maximal inequality for the non-semimartingale $u \mapsto L_1^0(W_u)$:

Theorem 5 (Kh–Révész–Shi '01) *As $\varepsilon \to 0$,*

$$\mathbb{P}\{\inf_{1\le u\le 2} L_1^0(W_u) \le \varepsilon\} \le e^{-\frac{1}{2}(1+o(1))\frac{|\log \varepsilon|}{\log|\log \varepsilon|}}.$$

From this, one can prove that

$$\lim_{u\to 0} L_1^0(W_u) = +\infty, \quad \text{almost surely.}$$

Open Problem 5* *Is there* γ *such that*

$$\ln \mathbb{P}\{\inf_{1\le u\le 2} L_1^0(W_u) < \varepsilon\} \sim \gamma \ln \varepsilon?$$

Is $\gamma > 1$*?*

There is not alot of room; indeed, it is easy to check that if so, $\gamma \geq 1$, since

$$\mathbb{P}\{\inf_{1\le u\le 2} L_1^0(W_u) \le \varepsilon\} \ge c\varepsilon.$$

There is something special about the local time at 0. Indeed, if we write $L_1^\star(W_u) = \sup_a L_1^a(W_u)$, one can easily show the following

Theorem 6 **(Kh–Révész–Shi '01)** *As $\varepsilon \to 0$,*

$$\ln \mathbb{P}\{\inf_{1 \le u \le 2} L_1^\star(W_u) < \varepsilon\} \sim -2j_1^2 \varepsilon^{-2},$$

where j_1 is the first positive zero of the modified Bessel function $\mathbf{J}_0$.

(Explosion of Local Times along Lines)

Here is an application to ergodic theory in Wiener space. Recall the ratio ergodic theorem for Brownian motion: $\forall f, g \in L^1(dx)$ with $\int g(x)\,dx \neq 0$, and for all $u \in [1,2]$,

$$\lim_{t\to\infty} \frac{\int_0^t f(W_u(v))\,dv}{\int_0^t g(W_u(v))\,dv} = \frac{\int f(x)\,dx}{\int g(x)\,dx}, \quad \text{a.s.}$$

In fact, this holds quasi-surely (Fitzsimmons '99): for all f, g given,

$$\lim_{t\to\infty} \frac{\int_0^t f(W_u(v))\,dv}{\int_0^t g(W_u(v))\,dv} = \frac{\int f(x)\,dx}{\int g(x)\,dx}, \quad \forall u \in [1,2], \text{ a.s.}$$

Note the order of the quantifiers! If f and g are slightly nicer than L^1, the above holds in a very strong sense:

Theorem 7 **(Kh–Révész–Shi '01)** *If* $f, g \in L^1(\{1+|x|\}dx)$ *and* $\int g(x)\, dx \neq 0$,

$$\lim_{t\to\infty} \sup_{1\le u\le 2} \left| \frac{\int_0^t f(W_u(s))\, ds}{\int_0^t g(W_u(s))\, ds} - \frac{\int f(x)\, dx}{\int g(x)\, dx} \right| = 0,$$

almost surely.

No rates seem possible, in general.

Open Problem 7* *Can the* $L^1(\{1+|x|\}dx)$ *condition be dropped to* $L^1(dx)$*?*

Next, we want a reason for

$$\mathbb{P}\{\inf_{1\le u\le 2} L_1^0(W_u) < \varepsilon\} \le e^{-c\frac{|\log\varepsilon|}{\log|\log\varepsilon|}}.$$

While I do not have a simple explanation for this, here is an attempt at a heuristic.

First, one makes rigorous the fact that no matter the value of u, if W_u makes alot of excursions from $(-\infty, 0)$ to $(0, \infty)$, then the chance that $L_1^0(W_u) < \varepsilon$ is very small. This is essentially a (conditional) large-deviations estimate, once one knows what alot really is. On the other hand, it is not hard to check that the LIL holds quasi-everywhere (Fukushima, Zimmerman, Walsh), i.e., with probability one:

$$\limsup_{v\to 0} \frac{W_u(v)}{\sqrt{2v \ln|\ln v|}} = 1, \qquad \forall u \in [1, 2].$$

Note the order of the quantifiers!

This suggests that no matter the value of u, W_u must make alot of excursions from $(-\infty, 0)$ to $(0, \infty)$. But how many? For this, we need to study Brownian sheet on three scales.

First, consider the time interval $[0, 1]$ (scale 1), and split it as $\cup_{k\geq 0}[\Phi_{k+1}^{k+1}, \Phi_k^k]$, (scale 2) where $\Phi_k = 32\varepsilon_0^{-2}k$ for a fairly explicit constant ε_0 (in terms of ε). It then turns out that all of the action is for the values $n \leq k \leq 2n$, where $n = |\ln \varepsilon| \div \ln|\ln \varepsilon|$. (scale 3)

An interval $\mathcal{J}_k = [\Phi_{k+1}^{k+1}, \Phi_k^k]$ is good if simultaneously over all $u \in [1, 2]$, $v \mapsto W_u(v)$ upcrosses or downcrosses $[-\varepsilon\Phi_{k+1}^{-k/2}, \varepsilon\Phi_k^{-k/2}]$ for some $t \in \mathcal{J}_k^\circ$ where $\mathcal{J}_k^\circ$ is an appropriately chosen interval in $\mathcal{J}_k$. Then, roughly speaking, $\mathcal{J}_k$'s are good in an independent fashion as k varies. So, by "large deviations", as k varies from n to $2n$, we expect n good intervals. Once you see a good interval, another large-deviations estimate shows that the local time will be large with "good" probability.