Intermittence & Multifractality

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The Stochastic Heat Equation on $\mathbb{R}$

- Consider SHE on $\mathbb{R}$: $\xi :=$ space-time white noise;

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  - The parabolic Anderson model (PAM): $\sigma(u) = u$ and $u(0\,,x) = 1$.

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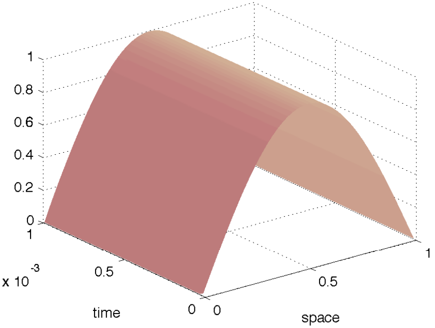
  - Nonlinear noise excitation ($\lambda \to \pm\infty$). Kim-K, 2014

The Stochastic Heat Equation on $[0\,,1]$

$\dot{u}(t\,,x) = u''(t\,,x) + \lambda\sigma(u(t\,,x))\xi(t\,,x)$ ***for*** $(t\,,x) \in (0\,,\infty) \times [0\,,1]$ ***with Dirichlet BC***
$u(0\,,x) = \sin(\pi x)$

$\boxed{\lambda = 0}$



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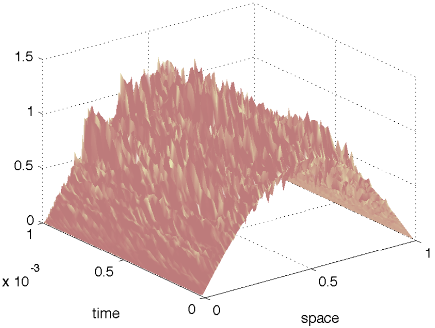

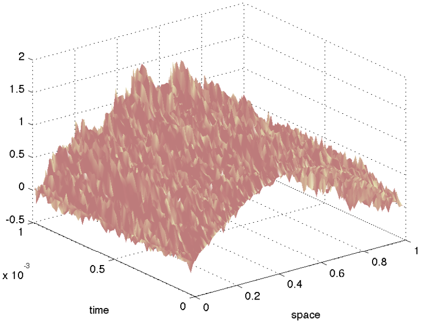

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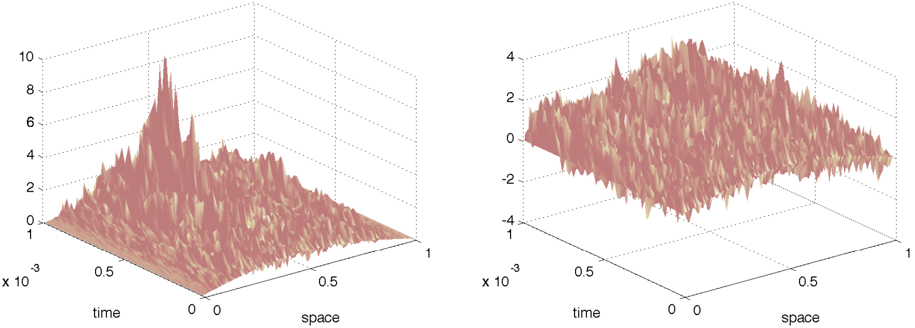

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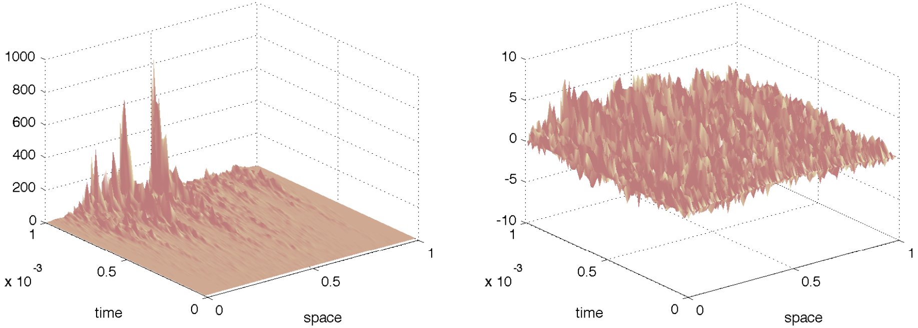
$$\boxed{\lambda = 10}$$



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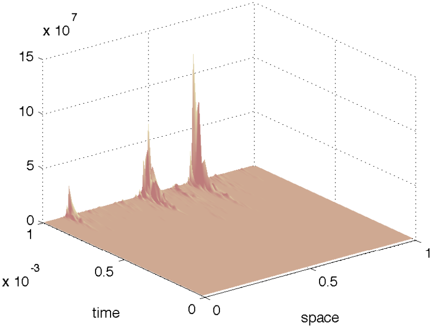

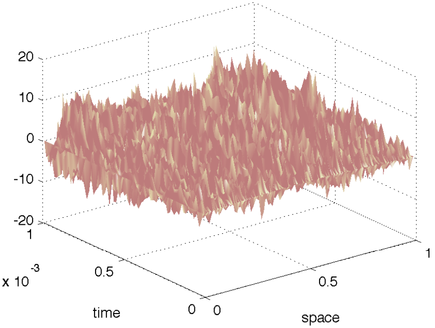

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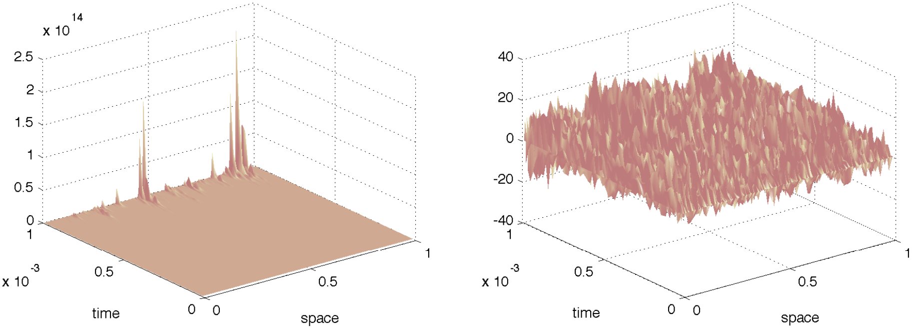
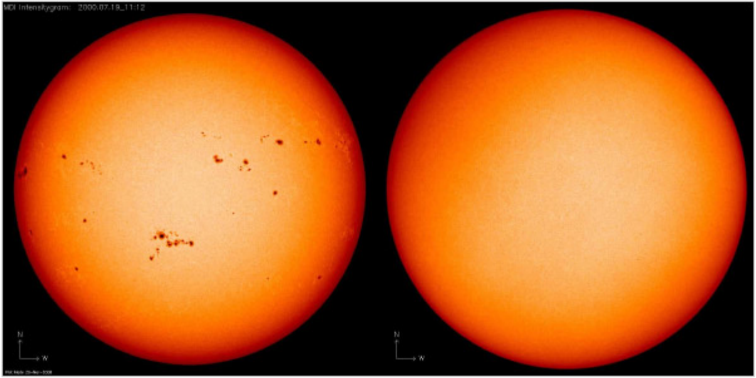


A Related Picture

Solar prominence video `http://apod.nasa.gov/apod/ap110307.html`

Is the Sun Missing Its Spots?



NASA

SUN GAZING These photos show sunspots near solar maximum on July 19, 2000, and near solar minimum on March 18, 2009. Some global warming skeptics speculate that the Sun may be on the verge of an extended slumber.

By KENNETH CHANG
Published: July 20, 2009

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where $\mathrm{Dim}_{_H} A < 0$ *means* A *is bounded.*

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- To simplify the exposition I will only talk about large-scale fractals in $[0\,,\infty)$ today.

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- Now define the large-scale Hausdorff dimension of A as

$$\text{Dim}_{\text{H}} A := \inf \left\{\rho > 0 : \sum_{n=0}^{\infty} \nu_n^{(\rho)}(A) < \infty\right\}.$$

Simple Facts About Large-Scale Hausdorff Dimension

$\nu_n^{(\rho)}(A) := \inf\{\sum_i (r_i/e^n)^\rho : \exists\{x_j\}$ ***s.t.*** $\cup_{i\geq 1} [x_i, x_i + r_i) \supset A \cap [e^n, e^{n+1})\}$

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  - $\therefore \inf_{n\geq 1} \nu_n^{(\rho)}(A) > 0$ if $\rho < 1 \Rightarrow \mathrm{Dim}_H \mathbb{N} \geq \rho$ for all $\rho < 1$. □

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- Choose and fix $c > q/\mathcal{D}_\mu(A)$, and then apply Frostman's lemma.

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- Remainder of today: Formulas for $\mathrm{Dim}_{\mathrm{H}}\, A$ where A is a non-trivial random set that is simpler to analyze than those in the SPDE examples earlier

Law of the Iterated Logarithm

$B :=$ **1-D Brownian motion,** $c > 0$

- Consider the random set $\mathcal{L}_c^B := \left\{ t \geq 8 : \; B(t) > c\sqrt{2t \log\log t} \right\}.$

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- Therefore $\text{Dim}_{\text{H}}\, \mathscr{L}_c^B = \begin{cases} 0 & \text{if } c > 1, \\ 1 & \text{if } c < 1. \end{cases}$ What about $\mathscr{L}_1^B$?

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- Apply Frostman to see that $\nu_n^{(1)}(\mathcal{L}_1^B) \geq cn^{-1}(\log n)^{-1/2}$ for most n's, a.s. Since $\sum_n n^{-1}(\log n)^{-1/2} = \infty$, we obtain the result. □

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  - The lower bound is slightly different from the preceding lower-bound methods …

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- One can show that such a [$\approx$ self-similar] set will have dimension $\ge 1-\rho$; therefore, $\text{Dim}_{\text{H}}\, \mathscr{L}_c^X \ge 1-\rho$ for all $\rho \in (c^2\,, 1)$. □