Dissipation and High Disorder

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A Brief Outline

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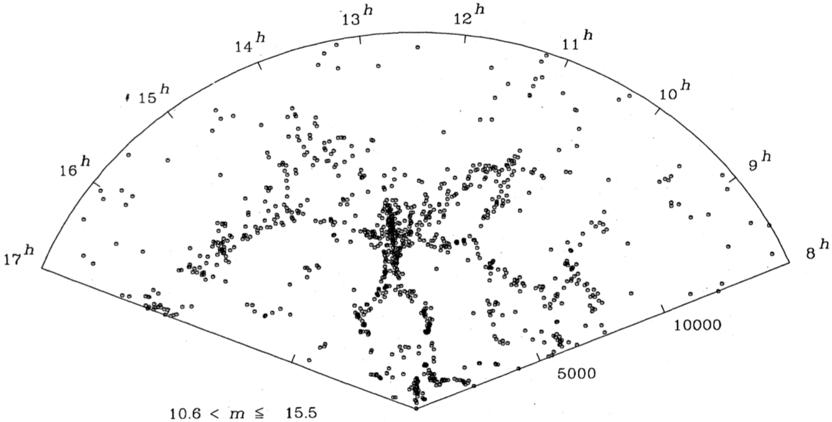
- Part of a big-picture analysis of intermittency & sensitivity of complex systems
- Connections to topics such as metastability & phase transition

Large-scale structure of galaxies

S. F. Shandarin and Ya B. Zeldovich, *Rev. Modern Phys. (1989)*



A simple model for intermittency

$[\dot{u}_t(x) = \frac{1}{2}u_t''(x) + \lambda u_t(x)\eta_t, u_0(x) = 1]$

(Zeldovich–Ruzmaikin–Sokoloff, 1990)

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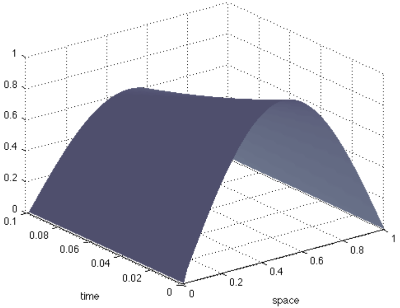
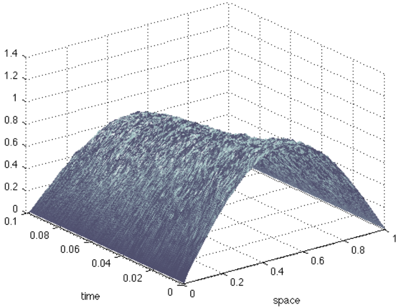
- $u_t \to 0$ as $\lambda \to \infty$
- $\mathrm{E}(u_t^2) = \exp\left\{\lambda^2 t\right\} \to \infty$ (fast!) as $\lambda \to \infty$

A SHE simulation $[\dot{u}_t(x) = \frac{1}{2}u_t''(x) + \lambda u_t(x)\eta_t(x)$, $u_0(x) = \sin(\pi x)$, $0 \leqslant x \leqslant 1$; $u_t(0) = u_t(1) = 0$.]

$\lambda = 0$ ***(left;*** $u_t(x) = \sin(\pi x)\exp(-\pi^2 t/2)$***)*** ***and*** $\lambda = 0.1$ ***(right)***





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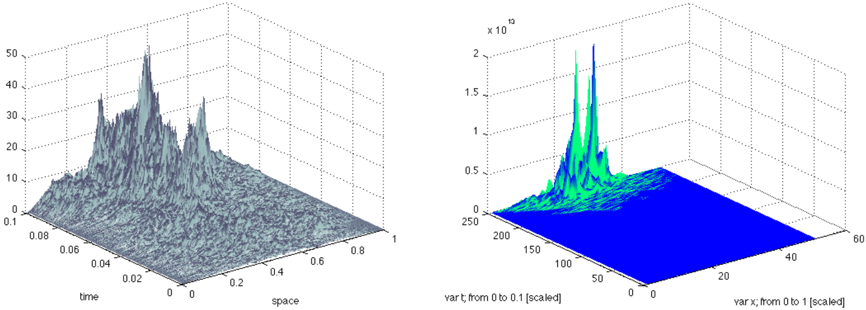
$\lambda = 2$ ***(left) and*** $\lambda = 6$ ***(right)***



Carmona-Molchanov Theory

- $du_t(x) = (\Delta_{\mathbf{Z}^d} u_t)(x)dt + \lambda u_t(x)dB_t(x)$, $x \in \mathbf{Z}^d$, $\lambda > 0$

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- *If $d = 1, 2$, then $\gamma_k > 0$.*
- *If $d \geq 3$, then $\exists k(\lambda)$ such that $\gamma_k > 0$ iff $k \geq k(\lambda)$.*

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- $\gamma_k > 0 \; \forall k \geq K$ *iff* $k \mapsto \gamma_k$ *is strictly increasing on* $[K, \infty)$.

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- I.e., If $\lambda, d \gg 1$ then many of the moments do not grow fast.

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    - Corollary: "Moment intermittency" [Foondun–K, 2009].

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$d \geq 1 \Rightarrow \exists\lambda_1 > 0 : \lambda > \lambda_1 \Rightarrow \lim_{t\to\infty} u_t(x) = 0$ *a.s.* [fast!] $\forall x \in \mathbf{Z}^d$.

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$d \geq 3 \Rightarrow \exists\lambda_2 > 0 :\ 0 < \lambda < \lambda_2 \ \Rightarrow\ \lim_{t\to\infty} \sup_{x\in\mathbf{Z}^d} u_t(x) = 0$ *a.s.*

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- I.e., Local dissipation is generic [also Greven-den Hollander, 2007]

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- Fact 2. $m(\lambda)$ is a nonnegative, mean-c_0, continuous martingale [Spitzer, 1980; Dawson–Perkins, 2012]. Here is the proof:

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    - Apply Duhamel's pcpl [Shiga and Shimizu, 1980]:

$$u_t(x) = c_0 p_t(-x) + \lambda \sum_{y\in\mathbf{Z}^d} \int_0^t p_{t-s}(y-x)\sigma(u_s(y))\, dB_s(y).$$

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  - Apply Duhamel's pcpl [Shiga and Shimizu, 1980]:

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- Is there a second phase point? [Probably not].

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where $a, b, \alpha, \gamma \in (0\,,\infty)$ are not germaine to the discussion.
- Conclude that

$$f(t) \le p \cdot \begin{cases} \exp\left(-qt^{1/3}\right) & \text{if } d = 1, \\ \exp\left(-r\sqrt{\log t}\right) & \text{if } d = 2. \end{cases}$$

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- Are there good lower bound? We can only prove that $\forall d \ge 1$ and $\lambda > 0$, $\mathrm{E}\sqrt{m_t(\lambda)} \ge G \cdot \exp(-Ht)$ as $t \to \infty$ a.s.

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- Therefore,

$$\begin{aligned}\lambda_c &:= \inf\{\lambda > 0 : \ m_\infty(\lambda) = 0 \text{ a.s.}\}\\ &= \sup\left\{\lambda > 0 : \ \mathrm{E}e^{-m_\infty(\lambda)} < 1\right\}\\ &= \inf\left\{\lambda > 0 : \ \mathrm{E}e^{-m_\infty(\lambda)} = 1\right\}\end{aligned}$$

$[\inf\varnothing := \infty]$ $\Rightarrow 0 \leq \lambda_c \leq \infty$ $\qquad [d = 1, 2 \Rightarrow \lambda_c = 0]$

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If $u_0 \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ is ≥ 0, then $\exists A, B \in (0\,,\infty)$ such that a.s.,

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- For eq's on compact sets, $\|u_t\|_{L^1} \leq Ae^{-Bt}$ a.s. …this is sharp.

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- $0 < \Lambda < \infty$ *iff a.s.* $\exists\tau \in (0\,,\infty)$ *such that* $\sup_{x\in\mathbf{R}} u_t(x) < \infty$ *for all* $0 < t < \tau$ *and* $\sup_{x\in\mathbf{R}} u_t(x) = \infty$ *for all* $t > \tau$.