An Asymptotic Theory for Randomly-Forced Heat Equations

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Outline

- The heat equation with random forcing

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- The linear equation and its connections with local times of LP's

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- The linear equation and its connections with local times of LP’s
- The nonlinear equation & intermittency, and their connections with recurrence/transience of LP’s

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  - What if $\dot{W}$ is replaced by spatially-colored noise?

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- $\therefore\ \hat{v}(t\,,\xi) = e^{-t\Psi(\xi)}$, and the solution is measure-valued:

$$v(t\,,A) := P\{X_t \in A\} := P_t(A).$$

The heat equation with forcing

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- Interpretation: Multiply by $\phi \in \mathscr{S}(\mathbf{R}_+ \times \mathbf{R}^d)$:

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- Solve by variation of parameters [Duhamel's formula].

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- Therefore (Dalang, 1999): the heat equation has function solutions iff $[1 + 2\mathrm{Re}\Psi]^{-1} \in L^1(\mathbf{R}^d)$.

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Let $\bar X_t := X_t - X_t'$ by the symmetrization of X.

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Suppose b is bounded and Lipschitz continuous, and the linear heat equation has a function solution u with $u(0\,,x) = 0$. Consider

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- Using local-time theory, we can construct u with $\text{Osc}\,u\equiv\infty$.
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- Everything holds if b is locally Lipschitz.

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- (Dalang, 1999): If the linear equation $[\sigma \equiv 0]$ has a unique solution, then the nonlinear one does too.

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Proposition (Carmona and Molchanov '94)
Intermittency holds if $\overline{\gamma}(2) > 0$ and $\overline{\gamma}(p) < \infty$ for all $p \geq 2$.

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- $\mathbf{R} = 1$, necessarily.

Theorem (Foondun-K, '09)

- *Suppose $\bar{X}_t := X_t - X'_t$ is recurrent and $\lim_{|x|\to\infty} |\sigma(x)/x| > 0$. Then there exists $\eta_0 > 0$ such that if $u(0\,,x) \geq \eta_0$ for all x, then u is intermittent.*

- Suppose the linear equation has a solution.
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Then, $u(t\,,\cdot) \in L^2(\mathbf{R}^d)$ a.s. for all $t > 0$, and

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Theorem (Dalang, '99; Nualart & Quer-Sardanyons, '06; Foondun-K, '10+)

If σ is Lipschitz and $u(0\,,\cdot)$ is nonrandom and bounded, then

$$\frac{\partial}{\partial t}u(t\,,x) = (Lu)(t\,,x) + \sigma(u(t\,,x))\dot{F}(t\,,x)$$

has a unique solution provided that

$$\int_{\mathbf{R}^d} \frac{\hat{f}(\xi)}{1 + 2Re\Psi(\xi)}\,d\xi < \infty \qquad \text{[iff, when } \sigma := const\text{]}.$$

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$$\|\exp(-(t/2)\mathrm{Re}\Psi)\|_2^2 = (2\pi)^d \|p_{t/4} * \check{p}_{t/4}\|_2^2$$

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  2. $p_t \exists$ and is in $L^\infty(\mathbf{R}^d)$ for all $t > 0$;
  3. $\exp(-t\mathrm{Re}\Psi) \in L^1(\mathbf{R}^d)$ for all $t > 0$.
- **Proof:** Since $\|p_t\|_2^2 \le \|p_t\|_\infty$, (2)⇒(1). Since $p_t = p_{t/2} * p_{t/2}$, Young's inequality: $\|p_t\|_\infty \le \|p_{t/2}\|_2^2$; therefore, (1)⇔(2). By the inversion theorem, (3)⇒(1)+(2). Suppose (1)+(2). Let $\check{f}(x) := f(-x)$. The F.T. of $p_{t/4} * \check{p}_{t/4}$ is $\exp(-(t/2)\mathrm{Re}\Psi)$. By Plancherel,

$$\|\exp(-(t/2)\mathrm{Re}\Psi)\|_2^2 = (2\pi)^d \|p_{t/4} * \check{p}_{t/4}\|_2^2 \le (2\pi)^d \|p_{t/4} * \check{p}_{t/4}\|_\infty.$$

$\therefore$ by Young's inequality,

$$\|\exp(-t\mathrm{Re}\Psi)\|_1 = \|\exp(-(t/2)\mathrm{Re}\Psi)\|_2^2 \le (2\pi)^d \|p_{t/4}\|_2^2.$$

$\therefore$ (1)+(2)⇒(3). $\square$

Concluding remarks

Lemma (Zabczyk, '70)

Suppose X is a radial Lévy process in $d \geq 2$. Then, $P_t(dx) \ll dx$ if and only if $\Psi(\xi) \to \infty$ as $\|\xi\| \to \infty$.

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- False if $d = 1$.
- What are good NASC conditions for $P_t(dx) \ll dx$ in terms of Ψ?