Additive Lévy Processes, II: A Proof, and More Recent Results

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Outline of Lecture 2:

- Intersections of Brownian Motions
  - Proof in the critical case $d = 4$
  - Proof in the subcritical case $d \leq 3$
- A problem on images

Intersections of Brownian Motions

In order to see how one can use multiparameter processes, let us isolate a concrete problem:

Theorem. [Dvoretzky et al (1950)] *Let* B *and* B' *be two independent BM's in* $\mathbf{R}^d$ *with* $B(0) = B'(0) = 0$*. Then* $B((0,\infty)) \cap B'((0,\infty)) \neq \varnothing$ *iff* $d \leq 3$*.*

The "usual" proof: By Kakutani (1944) [see also Dvoretzky et al (1950)]: For all nonrandom $G \subset \mathbf{R}^d$,

$$\mathrm{P}\Big\{B((0,\infty)) \cap G \neq \varnothing\Big\} > 0 \;\Leftrightarrow\; \mathrm{Cap}_{d-2}(G) > 0.$$

Therefore, it suffices to prove that

$$\mathrm{E}\Big[\overbrace{\mathrm{Cap}_{d-2}(B'((0,\infty)))}^{:=\mathrm{Cap}_{d-2}(G)}\Big] > 0 \;\Leftrightarrow\; d \leq 3.$$

And this is what most proofs do, after some fashion.

Alternatively, define A to be the additive Brownian motion,

$$A(s,t) := B(s) - B'(t).$$

And note that

$$B((0,\infty)) \cap B'((0,\infty)) \neq \varnothing \iff 0 \in A((0,\infty)^2).$$

So the problem is to prove that

A hits zero iff $d \leq 3$.

There is now a simple proof of this fact (Kh. 2003). The hardest part is $d = 4$ which I start with. [This proves also the case that $d \geq 5$, by projection.]

Proof in the Critical Case $d = 4$

Let $d = 4$, and $A(s\,,t) := B(s) - B'(t)$ be additive BM.

Step 1. $\mathrm{E}|A([0\,,1]^2)| = 0$.

Step 2. $0 \not\in A([1\,,2]^2)$.

Step 3. $0 \not\in A((0\,,\infty)^2)$ a.s.

Step 3 follows fairly easily from Steps 1 and 2, and scaling. Step 2 is also easy: Let

$$F(x) := \mathrm{P}\{x \in A([0\,,1]^2)\} \qquad {}^{\forall}x \in \mathbf{R}^4.$$

Step 1 $\Rightarrow F = 0$ a.e. $\Rightarrow$ because $A(1\,,1) \sim N(\mathbf{0}\,,4\mathbf{I})$,

$$\mathrm{P}\left\{0 \in A([1\,,2]^2)\right\} = (4\pi)^{-2} \int_{\mathbf{R}^4} F(x) e^{-\|x\|^2/4}\, dx = 0,$$

$\Rightarrow$ Step 2. Suffices to prove Step 1.

Recall $A(s,t) = B(s) - B'(t)$. We wish to show that $\mathrm{E}|A([0,1]^2)| = 0$ when $\boxed{d = 4}$.

Observation:

$$\mathrm{E}|A([0,2]^2)| \leq \mathrm{E}|A([0,1]\times[0,2])| + \mathrm{E}|A([1,2]\times[0,2])|.$$

By scaling, the left-hand side $= 4\mathrm{E}|A([0,1]^2)|$. The right-hand side terms are equal (stationarity of inc's). Thus,

$$2\mathrm{E}|A([0,1]^2)| \leq \mathrm{E}|A([0,1]\times[0,2])|.$$

$$\boxed{2\mathrm{E}|A([0,1]^2)| \leq \mathrm{E}|A([0,1]\times[0,2])|.}$$

But RHS =

$$\begin{aligned}
&\mathrm{E}|A([0,1]^2)| + \mathrm{E}|A([0,1]\times[1,2])| \\
&\quad - \mathrm{E}\left|A\Big([0,1]^2\Big) \cap A\Big([0,1]\times[1,2]\Big)\right| \\
&= 2\mathrm{E}|A([0,1]^2)| - \mathrm{E}\left|A\Big([0,1]^2\Big) \cap A\Big([0,1]\times[1,2]\Big)\right|.
\end{aligned}$$

Compare with the previous display:

$$\boxed{\mathrm{E}\left|A\Big([0,1]^2\Big) \cap A\Big([0,1]\times[1,2]\Big)\right| = 0.}$$

$$\mathrm{E}\left|A\Big([0,1]^2\Big)\cap A\Big([0,1]\times[1,2]\Big)\right| = 0$$

is the same as

$$\mathrm{E}\left(\underbrace{\left|\Big(B[0,1]-B'[0,1]\Big)\cap\Big(B[0,1]-B'[1,2]\Big)\right|}_{:=Z}\right) = 0$$

Thus for almost all $w \in \mathbf{R}^4$,

$$\mathrm{E}\left[Z \;\middle|\; B'(1)=w\right] = 0.$$

Given $\{B'(1)=w\}$, $B'[0,1]$ and $B'[1,2]$ are conditionally independent, both with the same law as the range of BM started at w. Because "Leb" is translation invariant,

$$\mathrm{E}\left|\Big(B[0,1]-B'[0,1]\Big)\cap\Big(B[0,1]-B''[0,1]\Big)\right| = 0.$$

Recall:

$$\mathrm{E}\left|\Big(B[0,1]-B'[0,1]\Big)\cap\Big(B[0,1]-B''[0,1]\Big)\right|=0.$$

Conditionally on $B[0,1]$, the two sets in $(\cdots)$ are i.i.d. copies of $A([0,1]^2)$. Therefore,

$$\mathrm{E}\left[\int_{\mathbf{R}^4}\left(\mathrm{P}\left\{x\in A\left([0,1]^2\right)\ \Big|\ B[0,1]\right\}\right)^2 dx\right]=0.$$

$\Rightarrow$ For almost-all $x\in\mathbf{R}^4$, $\mathrm{P}\{x\in A([0,1]^2)\}=0$

$\Rightarrow \mathrm{E}|A([0,1]^2)|=0$

$\Rightarrow$ Step 1.

Proof in the Subcritical Case $d \leq 3$

Proof á la Kahane (1983): Define

$$\sigma(F) := \iint_{\mathbf{R}^2_+} e^{-s-t} \mathbf{1}_F(A(s,t))\, ds\, dt.$$

Then $\sigma \in \mathscr{P}(A(\mathbf{R}^2_+))$ a.s., and

$$\hat{\sigma}(\xi) = \iint_{\mathbf{R}^2_+} e^{-s-t} e^{i\xi\cdot A(s,t)}\, ds\, dt.$$

Easy computation:

$$\mathrm{E}\left(\left|\hat{\sigma}(\xi)\right|^2\right) = \left(1 + \frac{\|\xi\|^2}{2}\right)^{-2}.$$

Therefore, when $d \leq 3$, $\mathrm{E}\|\hat{\sigma}\|^2_{L^2(\mathbf{R}^d)} < \infty$. By Plancherel, σ is a.s. absolutely continuous.

A Problem on Images

Let $\{X(t)\}_{t\geq 0}$ be a Lévy process in $\mathbf{R}^d$, and $F \subset \mathbf{R}_+$ a nonrandom compact set. Blumenthal and Getoor (1961) asked: (i) Is $\dim_{\rm H} X(F)$ a constant a.s.; and (ii) can one represent it in terms of the Lévy exponent Ψ? Kh. and Xiao (2005) proved, "Yes."

Define

$$\chi_\xi(x) := e^{-|x|\Psi(\mathsf{sgn}(x)\xi)} \qquad x \in \mathbf{R},\ \xi \in \mathbf{R}^d.$$

Also,

$$\mathscr{E}_\xi(\mu) := \iint \chi_\xi(x-y)\,\mu(dx)\,\mu(dy).$$

Then we have

Theorem. [Kh. and Xiao (2005)] *For all nonrandom Borel sets $F \subset \mathbf{R}_+$, $\dim_{\mathrm{H}} X(F)$ is a.s. equal to*

$$\sup\left\{\beta \in (0\,,d) : \inf_{\mu \in \mathscr{P}(F)} \int_{\mathbf{R}^d} \mathscr{E}_\xi(\mu) \frac{d\xi}{\|\xi\|^{d-\beta}} < \infty\right\}.$$

In the symmetric case this simplifies further. Define

$$f_\gamma(x) := \int_{\mathbf{R}^d} e^{-|x|\Psi(\xi)} \frac{d\xi}{\|\xi\|^\gamma} \qquad {}^\forall\gamma \in (0\,,d),\ x \in \mathbf{R}.$$

And

$$J_\gamma(\mu) := \iint f_\gamma(x-y)\,\mu(dx)\,\mu(dy).$$

Theorem. [Kh. and Xiao (2005)] *Suppose* X *is symmetric. For all nonrandom Borel sets* $F \subset \mathbf{R}_+$*,* $\dim_{\mathrm{H}} X(F)$ *is a.s. equal to*

$$\sup\left\{\beta \in (0\,,d):\ \inf_{\mu\in\mathscr{P}(F)} J_{d-\beta}(\mu) < \infty\right\}.$$

Useful Bounds

Recall

$$f_\gamma(x) := \int_{\mathbf{R}^d} e^{-|x|\Psi(\xi)} \frac{d\xi}{\|\xi\|^\gamma} \qquad {}^\forall \gamma \in (0\,,d),\ x \in \mathbf{R}.$$

Define

$$I(F) := \sup\left\{ \beta \in (0\,,d):\ \limsup_{r\to 0} \frac{\log f_{d-\beta}(r)}{\log(1/r)} < \dim_{\mathrm{H}} F \right\},$$

$$J(F) := \inf\left\{ \beta \in (0\,,d):\ \limsup_{r\to 0} \frac{\log f_{d-\beta}(r)}{\log(1/r)} > \dim_{\mathrm{H}} F \right\}.$$

Corollary. [Kh. and Xiao (2005)] *Suppose X is symmetric. For all nonrandom Borel sets $F \subset \mathbf{R}_+$,*

$$I(F) \le \dim_{\mathrm{H}} X(F) \le J(F).$$

Contours of ALPs

Now suppose $X_1, \ldots, X_N$ are independent symmetric Lévy processes in $\mathbf{R}^d$. Assume $\mathscr{X} := X_1 \oplus \cdots \oplus X_N$ is *absolutely continuous*:

$$\int_{\mathbf{R}^d} e^{-u\sum_{j=1}^N \Psi_j(\xi)}\, d\xi < \infty \quad {}^\forall u > 0.$$

Define the *gauge function*

$$\Phi(\boldsymbol{t}) := \int_{\mathbf{R}^d} e^{-\sum_{j=1}^N |t_j|\Psi_j(\xi)}\, d\xi \qquad {}^\forall \boldsymbol{t} \in \mathbf{R}^N_+.$$

Kh. and Xiao (2005) proved that

$$\mathrm{P}\left\{\mathscr{X}^{-1}(\{0\}) \neq \varnothing\right\} > 0 \;\Leftrightarrow\; \Phi \in L^1_{loc}(\mathbf{R}^N).$$

In fact, we know exactly when $\mathscr{X}^{-1}(\{0\}) \cap F \neq \varnothing$. There are also bounds on $\dim_{\mathrm{H}} \mathscr{X}^{-1}(\{0\})$ that held with positive probability(!).

Recall that

$$\Phi(\boldsymbol{t}) := \int_{\mathbf{R}^d} e^{-\sum_{j=1}^N |t_j| \Psi_j(\xi)}\, d\xi \qquad \forall \boldsymbol{t} \in \mathbf{R}_+^N.$$

Theorem. [Kh., Xiao, and Shieh, 2006] *Almost surely on* $\{\mathscr{X}^{-1}(\{0\}) \neq \varnothing\}$,

$$\dim_{\mathrm{H}} \mathscr{X}^{-1}(\{0\}) = \sup\left\{ q > 0 : \int_{[0,1]^d} \Phi(\boldsymbol{t}) \frac{d\boldsymbol{t}}{\|\boldsymbol{t}\|^q} < \infty \right\}.$$

There is a fairly explicit formula for $\dim_{\mathrm{H}} \mathscr{X}^{-1}(\{0\}) \cap F$, in fact.

Example 1

If $X_j =$ i.i.d. symmetric stable-$(\alpha_1, \ldots, \alpha_d)$. Then,

$$\Phi(\boldsymbol{t}) \asymp \|\boldsymbol{t}\|^{-\sum_{j=1}^{d}(1/\alpha_j)}.$$

Hence, $\Phi \in L^1_{loc}(\mathbf{R}^N) \Leftrightarrow \sum_{j=1}^{d}(1/\alpha_j) < N$. Therefore,

$$\mathrm{P}\left\{\mathscr{X}^{-1}(\{0\}) \neq \varnothing\right\} > 0 \ \Leftrightarrow \ \sum_{j=1}^{d} \frac{1}{\alpha_j} < N.$$

And a.s. on $\{\mathscr{X}^{-1}(\{0\}) \neq \varnothing\}$,

$$\dim_{\mathrm{H}} \mathscr{X}^{-1}(\{0\}) = \left(1 - \sum_{j=1}^{d} \frac{1}{\alpha_j}\right)_+.$$

Example 2

Suppose $X_j =$ iso. stable-α_j for $1 \le j \le N$. Without loss of generality, we may assume that

$$2 \ge \alpha_1 \ge \cdots \ge \alpha_N > 0.$$

Define

$$\kappa(\boldsymbol{\alpha}) := \min\left\{1 \le \ell \le N : \sum_{j=1}^{\ell} \alpha_j > d\right\},$$

where $\min\varnothing := \infty$. [$\kappa(\boldsymbol{\alpha}) = \infty$ iff $\sum_{j=1}^{N} \alpha_j \le d$.] Then:

$$\mathrm{P}\left\{\mathscr{X}^{-1}(\{0\}) \neq \varnothing\right\} > 0 \;\Leftrightarrow\; \kappa(\boldsymbol{\alpha}) < \infty.$$

If $\kappa(\boldsymbol{\alpha}) < \infty$, then a.s. on $\{\mathscr{X}^{-1}(\{0\}) \neq \varnothing\}$,

$$\dim_{\mathrm{H}} \mathscr{X}^{-1}(\{0\}) = N - \kappa(\boldsymbol{\alpha}) + \frac{1}{\alpha_{\kappa(\boldsymbol{\alpha})}}\left[\sum_{j=1}^{N} \alpha_j - d\right].$$