Additive Lévy Processes, I: Background and Motivation

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Outline of Lecture 1:

- Definition of LPs.
- Examples of LPs.
- Definition of ALPs.
- Do ALPs arise naturally?
- An application of ALPs to Lévy processes.
- Additive Brownian Motion and the Brownian Sheet [time permitting].

Definition of LPs

Let $X := \{X(t)\}_{t\geq 0}$ be a stoch. process [i.e., a sequence of rv's indexed by $\mathbf{R}_+$]. Suppose it takes values in $\mathbf{R}^d$; i.e., $X(t) \in \mathbf{R}^d$ with probab. one.

X is a *Lévy process* if:

- For all $t, s \geq 0$, $\{X(t+s) - X(s)\}_{t\geq 0}$ is [totally] independent of $\{X(u)\}_{0\leq u\leq s}$; ["*indep. incs*"]
- for all $t, s \geq 0$, $\{X(t+s) - X(s)\}_{t\geq 0}$ has the same [fi-di] distributions as $\{X(t)\}_{t\geq 0}$; ["*stat. inc's*"]
- $X(0) = 0$ and X is continuous in $L^0(\Omega, \mathscr{F}, \mathrm{P})$.

The distribution of the entire process X depends on the distribution of $X(t)$ which we realize via the Lévy–Khintchine formula for

$$\mathrm{E}\, e^{i\xi\cdot X(t)} = e^{-t\Psi(\xi)} \qquad \forall \xi \in \mathbf{R}^d, t \geq 0.$$

$\Psi :=$ Lévy exponent of X.

A Connection to Semigroups

Define for all $f : \mathbf{R}^d \to \mathbf{R}_+$ [Borel meas.], $x \in \mathbf{R}^d$, and $t \geq 0$,

$$(T_t f)(x) := \mathrm{E}\left[f(x + X(t))\right].$$

Let $\hat{f}(\xi) := \int_{\mathbf{R}^d} e^{i\xi\cdot x} f(x)\, dx$ and note that if $f, \hat{f} \in L^1(\mathbf{R}^d)$, then $f(x) = (2\pi)^{-d} \int_{\mathbf{R}^d} e^{i\xi\cdot x} \overline{\hat{f}(\xi)}\, d\xi$. Thus,

$$\begin{aligned}(T_t f)(x) &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \mathrm{E}\left[e^{i\xi\cdot X(t)}\right] e^{i\xi\cdot x} \overline{\hat{f}(\xi)}\, d\xi \\ &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-t\Psi(\xi)} e^{i\xi\cdot x} \overline{\hat{f}(\xi)}\, d\xi.\end{aligned}$$

Thus, T_t is a convolution kernel with multiplier $\widehat{T_t}(\xi) = e^{-t\Psi(\xi)}$.

$\{T_t\}_{t\geq 0}$ is a convolution semigroup.

Example: Brownian Motion

Think of $X : \mathbf{R}_+ \to \mathbf{R}^d$ as a “random function.”

- (Bachelier, 1900; Einstein, 1905) $\Psi(\xi) = \|\xi\|^2$.
- (Wiener, 1910) X is continuous a.s.
- (Paley, Wiener, Zygmund, 1933) X is nowhere-differentiable a.s.
- (Taylor, 1952/53) The random image-set $X(\mathbf{R}_+)$ a.s. has Hausdorff dimension $\min(d\,,2)$.
- The Hille–Yosida generator of $\{T_t\}_{t\geq 0}$ is Δ [distribution sense]; i.e., $T_t = e^{t\Delta}$.

Recall on Hausdorff Dimension

Let $s, \varepsilon > 0$ be fixed; $A \subset \mathbf{R}^d$ a set.

$$\mathscr{H}^s_\varepsilon(A) := \inf \sum_{j=1}^{\infty} (2r_j)^s,$$

where the infimum is taken over all balls $B_1, B_2, \ldots$ of respective radii $r_1, r_2, \ldots \in (0, \varepsilon)$ such that $\cup_{j=1}^{\infty} B_j \supset A$. The s*-dimensional Hausdorff measure* of A is

$$\mathscr{H}^s(A) := \lim_{\varepsilon \to 0^+} \mathscr{H}^s_\varepsilon(A).$$

$\mathscr{H}^s$ is an outer measure; measure on Borel sets.

$$\begin{aligned}\dim_{\mathrm{H}} A &:= \sup \{s > 0 : \ \mathscr{H}^s(A) > 0\} \\ &= \inf \{s > 0 : \ \mathscr{H}^s(A) < \infty\}.\end{aligned}$$

(Hausdorff, $\leq$1927)

A Relation to [Bessel-] Riesz Capacities

$$I_s(\mu) := \begin{cases} \iint \|x-y\|^{-s}\,\mu(dx)\,\mu(dy), & \text{if } s>0, \\ \iint \log_+ \|x-y\|^{-1}\,\mu(dx)\,\mu(dy), & \text{if } s=0, \\ 1, & \text{if } s<0. \end{cases}$$

$$\mathrm{Cap}_s(A) := \left[\inf_{\mu\in\mathscr{P}(A)} I_s(\mu) \right]^{-1}, \qquad [\inf\varnothing := \infty,\ 1/\infty := 0].$$

Theorem. [Frostman, 1935] *For all Borel sets $A \subset \mathbf{R}^d$,*

$$\begin{aligned} \dim_{\mathrm{H}} A &= \sup\{s>0:\ \mathrm{Cap}_s(A)>0\} \\ &= \inf\{s>0:\ \mathrm{Cap}_s(A)=0\}. \end{aligned}$$

Example: Isotropic Stable Processes

- (Lévy, 1937) $\Psi(\xi) = \|\xi\|^\alpha$; $\alpha \in (0,2]$.
- $T_t = \exp(t\Delta^{\alpha/2})$.
- (Lévy, 1937) When $\alpha < 2$, X is pure-jump a.s.
- (McKean, 1955) The random image-set $X(\mathbf{R}_+)$ a.s. has Hausdorff dimension $\min(d\,,\alpha)$.
- (Kakutani, 1944; Dvoretzky, Erdős, and Kakutani, 1950; McKean, 1955) For all Borel sets $A \subset \mathbf{R}^d$,

$$\mathrm{P}\left\{X(\mathbf{R}_+) \cap A \neq \varnothing\right\} > 0 \leftrightarrow \ \mathrm{Cap}_{d-\alpha}(A) > 0.$$

- (Nevanlinna, 1936; Noshiro, 1948; Ninomiya, 1953) Connections to the Dirichlet problem for $\Delta^{\alpha/2}$ with removable singularities.

Codimension and a Drawback

Recall that if X_α is iso. stable-α in $\mathbf{R}^d$ then

$$\mathrm{P}\{X_\alpha(\mathbf{R}_+) \cap A \neq \varnothing\} > 0 \leftrightarrow \ \mathrm{Cap}_{d-\alpha}(A) > 0.$$

Also recall (Frostman, 1935) that

$$\inf\{s \in (0\,, d): \ \mathrm{Cap}_{d-s}(A) > 0\} + \dim_{\mathrm{H}} A = d.$$

Thus,

Proposition. [Taylor, 1966] *For all Borel sets $A \subset \mathbf{R}^d$ with $\dim_{\mathrm{H}} A \geq d-2$,*

$$\inf\left\{\alpha \in (0\,,2] \;\middle|\; \mathrm{P}\{X_\alpha(\mathbf{R}_+) \cap A \neq \varnothing\} > 0\right\} + \dim_{\mathrm{H}} A = d.$$

What if $\dim_{\mathrm{H}} A < d-2$? An answer is given by Peres (1996; 1998), but this answer does not involve Lévy processes.

Additive Stable Processes

Let $X_1, \ldots, X_N$ denote independent iso. stable-α processes in $\mathbf{R}^d$. Define the (N,d)-random field

$$\mathscr{X}_{N,\alpha}(\boldsymbol{t}) := X_1(t_1) + \cdots + X_N(t_N), \qquad \boldsymbol{t} = (t_1, \ldots, t_N) \in \mathbf{R}^N_+.$$

["*additive stable process*"]

Theorem. [Hirsch–Song, 1995; Kh. 2002] *For all Borel sets $A \subset \mathbf{R}^d$,*

$$\mathrm{P}\left\{ \mathscr{X}_{N,\alpha}(\mathbf{R}^N_+) \cap A \neq \varnothing \right\} > 0 \leftrightarrow \mathrm{Cap}_{d-\alpha N}(A) > 0.$$

So now we can characterize $\dim_{\mathrm{H}} A$ by seeing for which pairs (N, α) the range of $\mathscr{X}_{N,\alpha}$ can hit A.

Definition of ALPs

- $X_1, \ldots, X_N =$ independent Lévy in $\mathbf{R}^d$;
- ALP [additive Lévy process]:

$$\boxed{\mathscr{X}(\boldsymbol{t}) := X_1(t_1) + \cdots + X_N(t_N),}$$

for $\boldsymbol{t} := (t_1, \ldots, t_N) \in \mathbf{R}^N_+$.

- Law is characterized by

$$\boxed{\mathrm{E}\, e^{i\xi\cdot\mathscr{X}(\boldsymbol{t})} = \exp\left(-\sum_{j=1}^N t_j \Psi_j(\xi)\right) = e^{-\boldsymbol{t}\cdot\boldsymbol{\Psi}(\xi)},}$$

where $\boldsymbol{\Psi}(\xi) := (\Psi_1(\xi), \ldots, \Psi_N(\xi))$ and Ψ_j is the Lévy exponent of X_j:

$$\mathrm{E}\, e^{i\xi\cdot X_j(s)} = e^{-s\Psi_j(\xi)}.$$

Do ALPs Arise Naturally?

Yes. Here are 4 ways; there are others as well.

1. Double Points

2. Triple Points, etc.

3. Arithmetic properties ["*Kahane's Problem*"]

4. Brownian sheet [time permitting]

Reason1: Intersections of Paths [“*Double Points*”]

Let Y be a Lévy process in $\mathbf{R}^d$. An old question:

When is $\mathscr{P} := \mathrm{P}\left\{ {}^{\exists} s \neq t : \; Y(s) = Y(t) \right\} > 0$?

- $Y =$ BM: $\mathscr{P} > 0$ iff $d \leq 3$

  (Dvoretzky, Erdős, and Kakutani, 1950; Aizenmann, 1985; Peres, 1996; Kh. 2003).

- Dvoretzky, Erdős, Kakutani, and Taylor (1957); Hendricks (1973/74); Hawkes (1977, 1978); Hendricks (1979); Kahane (1983, 1985); Evans (1987); Tongring (1988); Rogers (1989); Le Gall, Rosen, and Shieh (1989); Fitzsimmons and Salisbury (1989); Ren (1990); Hirsch and Song (1995); Shieh (1998); Peres (1999); Kh. (2002).

Connection to Additive Lévy Processes

When is $\mathscr{P} := \mathrm{P}\left\{ {}^{\exists} s \neq t : \; Y(s) = Y(t) \right\} > 0?$

Let Y_1 and Y_2 be i.i.d. copies. The above is equivalent to:

When is $\mathrm{P}\left\{ {}^{\exists} s,t > 0 : \; Y_1(s) = Y_2(t) \right\} > 0?$

Consider the additive Lévy process

$$\mathscr{Y}(\boldsymbol{t}) := Y_1(t_1) - Y_2(t_2).$$

We wish to know

When does $\mathscr{Y}$ hit zero?

Reason 2: Variants ["*Triple Points*"]

When is $\boxed{\mathrm{P}\left\{\exists \text{ distinct } s,t,u: \ Y(s)=Y(t)=Y(u)\right\} > 0?}$

If $Y = $ BM then the answer is "iff $d \leq 2$" (Dvoretzky, Erdős, Kakutani, and Taylor, 1957).

Equivalently, if Y_1, Y_2, Y_3 are i.i.d. Lévy processes then we wish to know when

$$\mathrm{P}\left\{\exists s,t,u > 0: \ Y_1(s)=Y_2(t)=Y_3(u)\right\} > 0?$$

Define $\mathscr{Y}(\boldsymbol{t}) := \begin{bmatrix} Y_1(t_1) \\ \mathbf{0}_d \\ \mathbf{0}_d \end{bmatrix} + \begin{bmatrix} \mathbf{0}_d \\ Y_2(t_2) \\ \mathbf{0}_d \end{bmatrix} + \begin{bmatrix} \mathbf{0}_d \\ \mathbf{0}_d \\ Y_3(t_3) \end{bmatrix}$. Then we wish to know when $\mathscr{Y}$ hits the diagonal of $\mathbf{R}^{3d}$; i.e., the collection of all points of the form $\boldsymbol{x} \otimes \boldsymbol{x} \otimes \boldsymbol{x}$ where $\boldsymbol{x} \in \mathbf{R}^d$.

Reason 3: Arithmetic Properties

- B, B_1, and $B_2 =$ indept. Br. motions in $\mathbf{R}^d$;
- $E, F \subset [0, \infty)$ compact, nonrandom, disjoint.

Then the following are equivalent (Kahane, 1985):

- $|B(E) \oplus B(F)| > 0$ with pos. probab.;
- $|B_1(E) \oplus B_2(F)| > 0$ with pos. probab. [does not require that $E \cap F = \varnothing$].

Kahane also provided a necessary as well as a sufficient condition, and asked for a precise condition on $E \times F$ to ensure $|B(E) \oplus B(F)| > 0$.

Answer: (Kh. 1999): $\boxed{\mathrm{Cap}_{d/2}(E \times F) > 0}$.

Define

$$\mathscr{B}(\boldsymbol{t}) := B_1(t_1) + B_2(t_2)$$

"*Additive Brownian motion*."

Then the following are equivalent:

- $|B_1(E) \oplus B_2(F)| > 0$ with pos. probab.
- $\mathscr{B}(E \times F)$ has positive Leb. meas. with pos. probab.

This problem is now very well understood for very general Lévy processes (Kh. and Xiao, 2005).

An Application of ALPs to LPs

Let Y be an arbitrary Lévy process in $\mathbf{R}^d$ with Lévy exponent Ψ_Y: $\mathrm{E}[e^{i\xi \cdot Y(s)}] = \exp(-s\Psi_Y(\xi))$. Our immediate goal is to derive the following:

Theorem. [Kh., Xiao, and Zhong, 2003] *With probability one,* $\dim_{\mathrm{H}} Y(\mathbf{R}_+)$ *is equal to*

$$\sup\left\{ s > 0 : \int_{\mathbf{R}^d} \mathrm{Re}\left(\frac{1}{1+\Psi_Y(\xi)} \right) \frac{d\xi}{\|\xi\|^{d-s}} < \infty \right\}.$$

This solved a relatively old problem (Taylor, 1952/53; McKean, 1955; Blumenthal and Getoor, 1960, 1961; Pruitt 1969; Fristedt, 1974).

$\Rightarrow$ If $\bar{Y} :=$ symmetrization of Y [i.e., $\bar{Y}(t) = Y(t) - Y'(t)$ for an indep. copy Y'] then $\dim_{\mathrm{H}} Y(\mathbf{R}_+) \geq \dim_{\mathrm{H}} \bar{Y}(\mathbf{R}_+)$ a.s. Completes the observation of Kesten (1969).

Let $X_1, \ldots, X_N$ be indept. Lévy processes in $\mathbf{R}^d$; $\Psi_j =$ Lévy exponent of X_j.

$$\mathscr{X}(\boldsymbol{t}) := X_1(t_1) + \cdots + X_N(t_N).$$

Theorem. [Kh. and Xiao 2006+] *Let $F \subset \mathbf{R}^d =$ nonrandom Borel. Then, we have $|\mathscr{X}(\mathbf{R}^N_+) \oplus F| > 0$ with pos. probab. iff $^\exists\mu \in \mathscr{P}(F)$:*

$$\int_{\mathbf{R}^d} \prod_{j=1}^{N} \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi)}\right) |\hat{\mu}(\xi)|^2 \, d\xi < \infty.$$

This improves on Kh., Xiao, and Zhong (2003).

When $N = 1$ this is well known (Orey, 1967; Kesten, 1969; Port and Stone, 1971; Hawkes, 1979, 1986; ...). For $N > 1$ we need different ideas.

For instance, suppose $X_1, \ldots, X_N$ are i.i.d., isotropic α-stable Lévy processes in $\mathbf{R}^d$. The "*additive stable process*" $\mathscr{X}_{N,\alpha}(\boldsymbol{t}) := \sum_{j=1}^N X_j(t_j)$ has the property that for all non-random analytic sets $F \subset \mathbf{R}^d$ the following are equivalent:

1. $\mathrm{P}\{|\mathscr{X}_{N,\alpha}(\mathbf{R}_+^N) \oplus F| > 0\} > 0$;

2. $^\exists \mu \in \mathscr{P}(F)$ such that

$$\int_{\mathbf{R}^d} \frac{|\hat{\mu}(\xi)|^2}{1 + \|\xi\|^{\alpha N}} \, d\xi < \infty;$$

3. $^\exists \mu \in \mathscr{P}(F)$ such that

$$I_{d-\alpha N}(\mu) = \iint \frac{\mu(dx)\,\mu(dy)}{\|x-y\|^{d-\alpha N}} < \infty;$$

4. $\mathrm{Cap}_{d-\alpha N}(F) > 0$.

We saw this before too.

I.e., if $X_1, \ldots, X_N$ are i.i.d. isotropic stable-α in $\mathbf{R}^d$ and $\mathscr{X}_{N,\alpha}(\boldsymbol{t}) = X_1(t_1) + \cdots + X_N(t_N)$, then

$$\mathrm{P}\left\{ \left| \mathscr{X}_{N,\alpha}(\mathbf{R}_+^N) \oplus F \right| > 0 \right\} > 0 \;\leftrightarrow\; \mathrm{Cap}_{d-\alpha N}(F) > 0.$$

Let Y be an independent [ordinary] Lévy process in $\mathbf{R}^d$, and apply the preceding to $F := Y(\mathbf{R}_+)$, conditional on Y: As positive-probability events,

$$\boxed{\left| \mathscr{X}_{N,\alpha}(\mathbf{R}_+^N) \oplus Y(\mathbf{R}_+) \right| > 0 \;\leftrightarrow\; \mathrm{Cap}_{d-\alpha N}(Y(\mathbf{R}_+)) > 0.}$$

But $\mathscr{X}_{N,\alpha}(\mathbf{R}_+^N) \oplus Y(\mathbf{R}_+) = \mathscr{Z}(\mathbf{R}_+^{N+1})$, where $\mathscr{Z}$ is the ALP $\mathscr{Z}(\boldsymbol{t}) := X_1(t_1) + \cdots + X_N(t_N) + Y(t_{N+1})$. We can apply the previous theorem with $F := \{0\}$ to find that

$$\boxed{\left| \mathscr{Z}(\mathbf{R}_+^{N+1}) \right| > 0 \leftrightarrow \int_{\mathbf{R}^d} \mathrm{Re}\left(\frac{1}{1 + \Psi_{Y}(\xi)} \right) \frac{d\xi}{\|\xi\|^{\alpha N}} < \infty.}$$

Compare the boxed equations to find $\dim_{\mathrm{H}} Y(\mathbf{R}_+)$ (Frostman, 1935).

Additive BM and the Brownian Sheet

Let $B(t_1, t_2)$ denote two-parameter Brownian sheet in $\mathbf{R}^d$. I.e., B is a centered Gaussian process and

$$\mathrm{Cov}\left(B_i(s_1, s_2)\,, B_j(t_1, t_2)\right) = \min(s_1, t_1)\min(s_2, t_2)\delta_{i,j}.$$

Locally, Brownian sheet looks like additive Brownian motion. Here is a precise statement near the point $\boldsymbol{t} := (1\,, 1)$ (say!): For $\varepsilon, \delta > 0$,

$$B(1+\varepsilon\,, 1+\delta) = B(1\,, 1) + \underbrace{X_1(\varepsilon) + X_2(\delta)}_{\text{ABM}\approx\sqrt{\varepsilon+\delta}} + \underbrace{Y(\varepsilon\,, \delta)}_{\approx\sqrt{\varepsilon\delta}},$$

where:

- X_1 and X_2 are Brownian motions;
- Y is a Brownian sheet;
- $B(1\,, 1)$, X_1, X_2, and Y are totally independent.

When $\varepsilon, \delta \approx 0$,

$$B(1+\varepsilon, 1+\delta) - B(1,1) \approx X_1(\varepsilon) + X_2(\delta).$$

One makes precise sense of this in a problem-dependent manner. Here is an application:

Theorem. [Kh. and Shi, 1999] *If $F \subset \mathbf{R}^d$ is analytic, then $\mathrm{P}\{B((0,\infty)^2) \cap F \neq \varnothing\} > 0$ iff $\mathrm{Cap}_{d-4}(F) > 0$.*

[$\mathrm{Cap}_0 :=$ log. capacity; $\mathrm{Cap}_{-r} := 1$ for $r < 0$]

For instance, let $F := \{0\}$ to find that B hits singletons iff $d \leq 3$ (Orey and Pruitt, 1973).

By Girsanov this theorem translates to a statement about SPDEs:

Let $\dot{W}$ denote d-dimensional white noise over $\mathbf{R}^2$. Consider the system of SPDEs of the wave type,

$$\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = b(u(t,x)) + \mathbf{\Sigma}\dot{W}(t,x).$$

Here, $\mathbf{\Sigma}$ is a non-singular, $d \times d$ matrix, and $b : \mathbf{R}^d \to \mathbf{R}^d$ is bounded and globally Lipschitz (say!). Note that $u : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}^d$. Then, for all nonrandom analytic sets $F \subset \mathbf{R}^d$,

$$\boxed{\mathrm{P}\left\{u((0,\infty) \times \mathbf{R}_+) \cap F \neq \varnothing\right\} > 0 \leftrightarrow \ \mathrm{Cap}_{d-4}(F) > 0.}$$

A remarkable recent theorem of Robert Dalang and Eulalia Nualart (2004) shows that the same result holds for the system of SPDEs,

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 u(t,x)}{\partial x^2} + b(u(t,x)) + a(u(t,x))\dot{W}(t,x),$$

as long as a and b are Lipschitz and bounded (say!), and a is strictly elliptic.