# Roots and Chaos

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4. Is the model any good?

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3. Might draw conclusions about the existence of random patterns in various disciplines.

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- How is this done? First, a few facts about $\sqrt{2}$

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- Open Question: How many zeros in the decimal expansion of $\sqrt{2}$?
- Conjecture: [Émile Borel, *Comptes Rendus Acad Sci Paris* **230**, 1950, pp. 591–593] all digits 0–9 are equidistributed in the decimal expansion of $\sqrt{2}$

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# Mathematical Experimentation

$\sqrt{2} \approx 1.4142135623730950488016887242096980785696718753769480731766797379 \cdots$

- What does $\sqrt{2}$ "look like"? Here is a simple first attempt:

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- So $\sqrt{2} \approx 1.??$. Therefore:

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| 1.1 | 1.21 |
| 1.2 | 1.44 |
| 1.3 | 1.69 |
| 1.4 | 1.96 |
| $\sqrt{2}$ | 2 |
| 1.5 | 2.25 |

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- ...So $\sqrt{2} \approx 1.41??$. Therefore:

| $\mathbf{a}$ | $\mathbf{a}^2$ |
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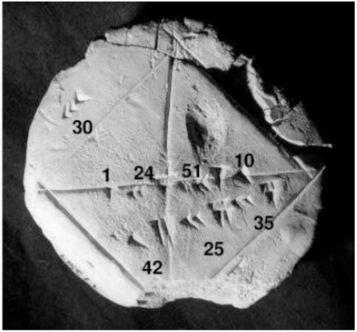
| 1.411 | 1.990921 |
| 1.412 | 1.993744 |
| 1.413 | 1.996569 |
| 1.414 | 1.999396 |
| $\sqrt{2}$ | 2 |
| 1.415 | 2.002225 |

$\Rightarrow 1.414 < \sqrt{2} < 1.415$ etc. ...

# The Babylonian Clay Tablet

(circa 1600–1800 BC; Yale Collection 7289)

$$\sqrt{2} \approx 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = \underbrace{1.41421}_{\checkmark}296296\cdots$$



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- **Theorem.** The preceding can be carried out *ad infinitum*; that is,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

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## Related Exercises

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- E.g., $x = \sqrt{5}$ can be written as $x^2 - 1 = 4 \Leftrightarrow (x-1)(x+1) = 4 \Leftrightarrow x = 1 + \frac{4}{1+x}$ ...
- This yields

$$\sqrt{5} = 1 + \cfrac{4}{2 + \cfrac{4}{2 + \cfrac{4}{2 + \cdots}}}$$

- Or $x^2 - 4 = 1 \Leftrightarrow (x-2)(x+2) = 1 \Leftrightarrow x = 2 + \frac{1}{2+x}$.

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- Do see a pattern?

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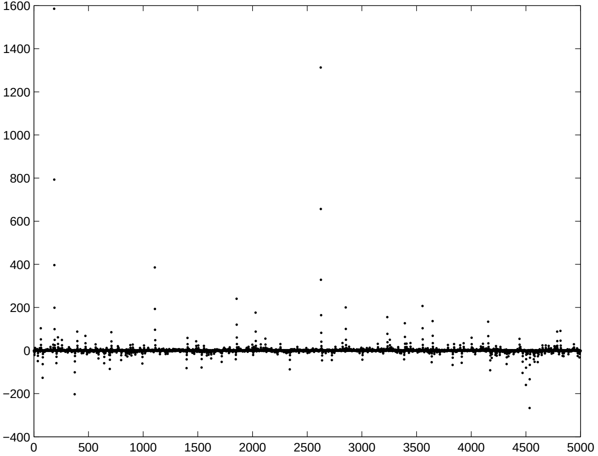
$x^2 = -1 \ldots x = \frac{x}{2} - \frac{1}{2x}$ . . . 5000 iterations



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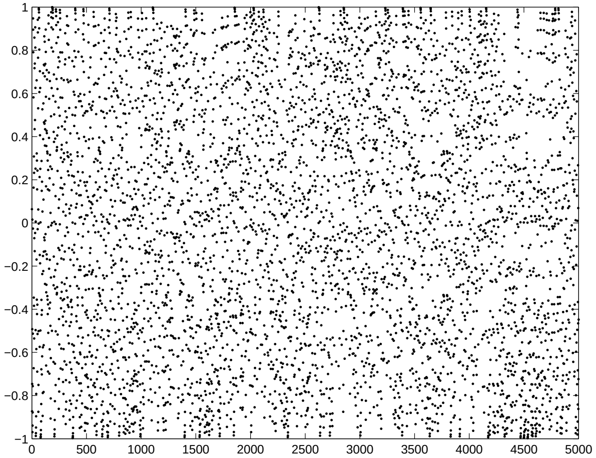
$x^2 = -1$ . . . $x = \frac{x}{2} - \frac{1}{2x}$ . . . 5000 iterations . . . "heavy tails" . . . $n$ versus $\frac{2}{\pi}\arctan(x_n)$



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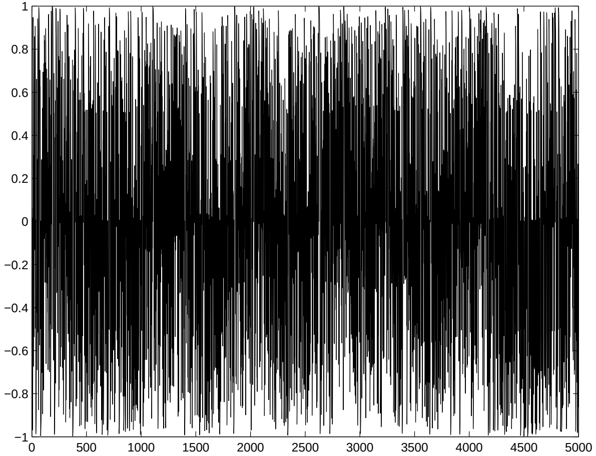

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