Tutorial on Additive Lévy Processes

Lecture #2

Davar Khoshnevisan

Department of Mathematics
University of Utah
http://www.math.utah.edu/˜davar

International Conference on Stochastic Analysis and Its Applications
August 7–11, 2006

Hausdorff Measure and Dimension

If $A \in \mathbf{R}^d$ and $s > 0$ then

$$H^s_\epsilon(A) := \inf\left\{ \sum_{n=1}^\infty (2r_n)^s : \ A \subset \bigcup_{n=1}^\infty B(x_n\,, r_n),\ 0 \le r_n \le \epsilon \right\}.$$

Hausdorff Measure and Dimension

If $A \in \mathbf{R}^d$ and $s > 0$ then

$$H^s_\epsilon(A) := \inf\left\{ \sum_{n=1}^{\infty} (2r_n)^s : A \subset \bigcup_{n=1}^{\infty} B(x_n, r_n),\ 0 \le r_n \le \epsilon \right\}.$$

Definition (Hausdorff, 1919)

The s-dimensional Hausdorff measure of A is

$$H^s(A) := \lim_{\epsilon \downarrow 0} H^s_\epsilon(A).$$

Spherical measure (Besicovitch)

Hausdorff Measure and Dimension

Theorem (Hausdorff, 1919)

For all $s > 0$, the restriction of H^s to Borel sets in $\mathbf{R}^d$ is a measure.

Hausdorff Measure and Dimension

Theorem (Hausdorff, 1919)

For all $s > 0$, the restriction of H^s to Borel sets in $\mathbf{R}^d$ is a measure.

Definition ("Hausdorff Dimension"; Hausdorff, 1919)

$$\dim_{\mathrm{H}} A = \sup\{s : H^s(A) = \infty\} = \inf\{s > 0 : H^s(A) = 0\}.$$

Hausdorff Measure and Dimension

Definition (s-dimensional energy of $\mu \in \mathscr{P}(A)$; M. Riesz)

$$I_s(\mu) := \iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^s}\ (s > 0),$$

Hausdorff Measure and Dimension

Definition (s-dimensional energy of $\mu \in \mathscr{P}(A)$; M. Riesz)

$$I_s(\mu) := \iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^s}\ (s>0),$$

Hausdorff Measure and Dimension

Definition (s-dimensional energy of $\mu \in \mathscr{P}(A)$; M. Riesz)

$$I_s(\mu) := \iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^s}\ (s>0), \quad I_0(\mu) := \iint \log_+ |x-y|^{-1} \mu(dx)\,\mu(dy).$$

Hausdorff Measure and Dimension

Definition (s-dimensional energy of $\mu \in \mathscr{P}(A)$; M. Riesz)

$$I_s(\mu) := \iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^s}_{(s>0)}, \; I_0(\mu) := \iint \log_+ |x-y|^{-1} \mu(dx)\,\mu(dy).$$

Definition (s-dimensional capacity of A; C. Gauss, M. Riesz)

$$\mathscr{C}_s(A) := \left[\inf_{\mu \in \mathscr{P}(A)} I_s(\mu) \right]^{-1}, \quad \inf \varnothing := \infty, \; 1/\infty := 0.$$

Hausdorff Measure and Dimension

Definition (s-dimensional energy of $\mu \in \mathscr{P}(A)$; M. Riesz)

$$I_s(\mu) := \iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^s}\ {\scriptstyle (s>0)},\quad I_0(\mu) := \iint \log_+ |x-y|^{-1}\mu(dx)\,\mu(dy).$$

Definition (s-dimensional capacity of A; C. Gauss, M. Riesz)

$$\mathscr{C}_s(A) := \left[\inf_{\mu\in\mathscr{P}(A)} I_s(\mu)\right]^{-1},\quad \inf\varnothing := \infty,\ 1/\infty := 0.$$

Men are liars. We'll lie about lying if we have to. I'm an algebra liar. I figure two good lies make a positive.

–Tim Allen

Hausdorff Measure and Dimension

Theorem (Frostman, 1935)

$$\dim_{\mathrm{H}} A = \sup\{s > 0 : \mathscr{C}_s(A) > 0\} = \inf\{s > 0 : \mathscr{C}_s(A) = 0\}.$$

Hausdorff Measure and Dimension

Theorem (Frostman, 1935)

$$\dim_{\mathrm{H}} A = \sup\{s > 0 : \mathscr{C}_s(A) > 0\} = \inf\{s > 0 : \mathscr{C}_s(A) = 0\}.$$

- If $\exists \mu \in \mathscr{P}(A)$ and $s > 0$ such that $I_s(\mu) = \iint |x - y|^{-s}\, \mu(dx)\, \mu(dy) < \infty$ then $\dim_{\mathrm{H}} A \geq s$.

Hausdorff Measure and Dimension

Theorem (Frostman, 1935)

$$\dim_{\mathrm{H}} A = \sup\{s > 0 : \mathscr{C}_s(A) > 0\} = \inf\{s > 0 : \mathscr{C}_s(A) = 0\}.$$

- If $\exists \mu \in \mathscr{P}(A)$ and $s > 0$ such that $I_s(\mu) = \iint |x-y|^{-s}\,\mu(dx)\,\mu(dy) < \infty$ then $\dim_{\mathrm{H}} A \geq s$.
- Formally, $\mathscr{C}_s(A) := 1$ if $s < 0$.

Hausdorff Measure and Dimension

If $x \in [0\,,1]$ then $x = \sum_{j=1}^{\infty} x_j 3^{-j}$ where $x_j \in \{0\,,1\,,2\}$ [ignore triadic rationals].

Hausdorff Measure and Dimension

If $x \in [0\,,1]$ then $x = \sum_{j=1}^{\infty} x_j 3^{-j}$ where $x_j \in \{0\,,1\,,2\}$ [ignore triadic rationals].

Example (Hausdorff, 1919)

Then $\dim_{\mathrm{H}} C = \log_3 2$ where $C = \{x \in [0\,,1] : x_j = 0 \text{ or } 2\}$.

Hausdorff Measure and Dimension

If $x \in [0\,,1]$ then $x = \sum_{j=1}^{\infty} x_j 3^{-j}$ where $x_j \in \{0\,,1\,,2\}$ [ignore triadic rationals].

Example (Hausdorff, 1919)

Then $\dim_{\mathrm{H}} C = \log_3 2$ where $C = \{x \in [0\,,1] : x_j = 0 \text{ or } 2\}$.

Hausdorff Measure and Dimension

If $x \in [0,1]$ then $x = \sum_{j=1}^{\infty} x_j 3^{-j}$ where $x_j \in \{0,1,2\}$ [ignore triadic rationals].

Example (Hausdorff, 1919)

Then $\dim_{\rm H} C = \log_3 2$ where $C = \{x \in [0,1] : x_j = 0 \text{ or } 2\}$.

Proof of upper bound: Cover C with 2^n intervals of length $2/3^n$.

$$\Rightarrow \; H^s_{2^{-n}}(C) \le 2^n \times (2^s 3^{-ns}) \to 0 \text{ if } s > \log_3 2.$$

$\Rightarrow \dim_{\rm H} C \le \log_3 2$.

Hausdorff Measure and Dimension

Proof of Lower Bound:
Let $X_1, X_2, \ldots$ be i.i.d. $P\{X_1 = 0\} = P\{X_1 = 2\} = 1/2$.

Hausdorff Measure and Dimension

Proof of Lower Bound:
Let $X_1, X_2, \ldots$ be i.i.d. $P\{X_1 = 0\} = P\{X_1 = 2\} = 1/2$.

$$X := \sum_{j=1}^{\infty} \frac{X_j}{3^j} \in C \qquad \text{“uniform}(C)\text{ distribution” (Cantor, Lebesgue)}$$

Hausdorff Measure and Dimension

Proof of Lower Bound:
Let $X_1, X_2, \ldots$ be i.i.d. $P\{X_1 = 0\} = P\{X_1 = 2\} = 1/2$.

$$X := \sum_{j=1}^{\infty} \frac{X_j}{3^j} \in C \qquad \text{“uniform}(C)\text{ distribution” (Cantor, Lebesgue)}$$

$$I_s(\mu) = \iint \frac{\mu(dx)\mu(dy)}{|x-y|^s} = E|X-Y|^{-s} \quad X \text{ and } Y \text{ iid copies}$$

Hausdorff Measure and Dimension

Proof of Lower Bound:
Let $X_1, X_2, \ldots$ be i.i.d. $P\{X_1 = 0\} = P\{X_1 = 2\} = 1/2$.

$$X := \sum_{j=1}^{\infty} \frac{X_j}{3^j} \in C \qquad \text{“uniform}(C)\text{ distribution” (Cantor, Lebesgue)}$$

$$\begin{aligned} I_s(\mu) &= \iint \frac{\mu(dx)\mu(dy)}{|x-y|^s} = E|X-Y|^{-s} \quad X \text{ and } Y \text{ iid copies} \\ &\leq E3^{Ns}, \end{aligned}$$

Hausdorff Measure and Dimension

Proof of Lower Bound:
Let $X_1, X_2, \ldots$ be i.i.d. $P\{X_1 = 0\} = P\{X_1 = 2\} = 1/2$.

$$X := \sum_{j=1}^{\infty} \frac{X_j}{3^j} \in C \qquad \text{“uniform}(C)\text{ distribution” (Cantor, Lebesgue)}$$

$$\begin{aligned} I_s(\mu) &= \iint \frac{\mu(dx)\mu(dy)}{|x-y|^s} = E|X-Y|^{-s} \quad X \text{ and } Y \text{ iid copies} \\ &\le E3^{Ns}, \quad N := \min\{n : X_n \neq Y_n\} = \text{Geom}(1/2). \end{aligned}$$

Hausdorff Measure and Dimension

Proof of Lower Bound:

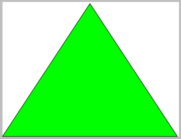
Let $X_1, X_2, \ldots$ be i.i.d. $P\{X_1 = 0\} = P\{X_1 = 2\} = 1/2$.

$$X := \sum_{j=1}^{\infty} \frac{X_j}{3^j} \in C \qquad \text{“uniform}(C)\text{ distribution” (Cantor, Lebesgue)}$$

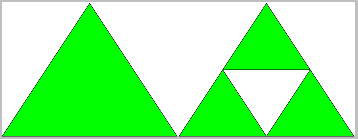
$$\begin{aligned} I_s(\mu) &= \iint \frac{\mu(dx)\mu(dy)}{|x-y|^s} = E|X-Y|^{-s} \quad X \text{ and } Y \text{ iid copies} \\ &\leq E3^{Ns}, \quad N := \min\{n : X_n \neq Y_n\} = \text{Geom}(1/2). \end{aligned}$$

$\Rightarrow I_s(\mu) < \infty$ if $s < \log_3 2$.

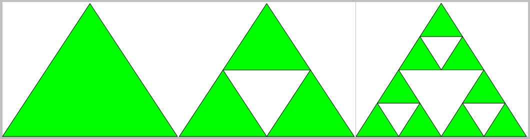
Sierpinski's Triangle △



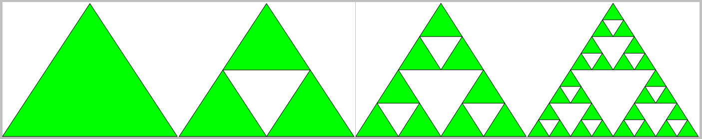
Sierpinski's Triangle △



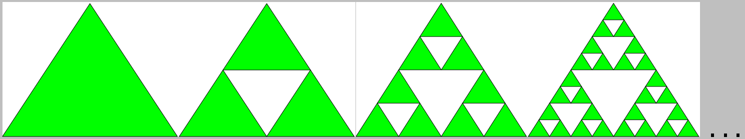
Sierpinski's Triangle △



Sierpinski's Triangle △

 …

Sierpinski's Triangle $\triangle$



Exercise

Compute $\dim_{\mathrm{H}} \triangle$

Stable Processes

Let X_α be a symmetric stable process in $\mathbf{R}^d$ with index $\alpha \in (0\,,2]$; i.e.,

- $E\exp(i\xi \cdot X_\alpha(t)) = \exp(-t|\xi|^\alpha/2)$ for all $\xi \in \mathbf{R}^d$;

Stable Processes

Let X_α be a symmetric stable process in $\mathbf{R}^d$ with index $\alpha \in (0\,,2]$; i.e.,

- $E\exp(i\xi \cdot X_\alpha(t)) = \exp(-t|\xi|^\alpha/2)$ for all $\xi \in \mathbf{R}^d$;
- $\{X_\alpha(t+s) - X_\alpha(s)\}_{t\geq 0}$ is independent of $\{X_\alpha(u)\}_{0\leq u\leq s}$;

Stable Processes

Let X_α be a symmetric stable process in $\mathbf{R}^d$ with index $\alpha \in (0\,,2]$; i.e.,

- $E\exp(i\xi \cdot X_\alpha(t)) = \exp(-t|\xi|^\alpha/2)$ for all $\xi \in \mathbf{R}^d$;
- $\{X_\alpha(t+s) - X_\alpha(s)\}_{t\geq 0}$ is independent of $\{X_\alpha(u)\}_{0\leq u\leq s}$;
- $\{X_\alpha(t+s) - X_\alpha(s)\}_{t\geq 0}$ has the same f.d.d.'s are $\{X_\alpha(u)\}_{u\geq 0}$;

Stable Processes

Let X_α be a symmetric stable process in $\mathbf{R}^d$ with index $\alpha \in (0\,,2]$; i.e.,

- $E \exp(i\xi \cdot X_\alpha(t)) = \exp(-t|\xi|^\alpha/2)$ for all $\xi \in \mathbf{R}^d$;
- $\{X_\alpha(t+s) - X_\alpha(s)\}_{t\geq 0}$ is independent of $\{X_\alpha(u)\}_{0\leq u\leq s}$;
- $\{X_\alpha(t+s) - X_\alpha(s)\}_{t\geq 0}$ has the same f.d.d.'s are $\{X_\alpha(u)\}_{u\geq 0}$;
- $P\{|X_\alpha(t) - X_\alpha(s)| \geq \epsilon\} \to 0$ as $t \to s$ for all $\epsilon > 0$;

Stable Processes

Let X_α be a symmetric stable process in $\mathbf{R}^d$ with index $\alpha \in (0\,,2]$; i.e.,

- $E \exp(i\xi \cdot X_\alpha(t)) = \exp(-t|\xi|^\alpha/2)$ for all $\xi \in \mathbf{R}^d$;
- $\{X_\alpha(t+s) - X_\alpha(s)\}_{t\geq 0}$ is independent of $\{X_\alpha(u)\}_{0\leq u\leq s}$;
- $\{X_\alpha(t+s) - X_\alpha(s)\}_{t\geq 0}$ has the same f.d.d.'s are $\{X_\alpha(u)\}_{u\geq 0}$;
- $P\{|X_\alpha(t) - X_\alpha(s)| \geq \epsilon\} \to 0$ as $t \to s$ for all $\epsilon > 0$;
- $t \mapsto X_\alpha(t)$ can be—and will be—assumed to be right-continuous with left-limits everywhere [discontinuities of the first kind only];

Stable Processes

Let X_α be a symmetric stable process in $\mathbf{R}^d$ with index $\alpha \in (0\,,2]$; i.e.,

- $E \exp(i\xi \cdot X_\alpha(t)) = \exp(-t|\xi|^\alpha/2)$ for all $\xi \in \mathbf{R}^d$;
- $\{X_\alpha(t+s) - X_\alpha(s)\}_{t\geq 0}$ is independent of $\{X_\alpha(u)\}_{0\leq u\leq s}$;
- $\{X_\alpha(t+s) - X_\alpha(s)\}_{t\geq 0}$ has the same f.d.d.'s are $\{X_\alpha(u)\}_{u\geq 0}$;
- $P\{|X_\alpha(t) - X_\alpha(s)| \geq \epsilon\} \to 0$ as $t \to s$ for all $\epsilon > 0$;
- $t \mapsto X_\alpha(t)$ can be—and will be—assumed to be right-continuous with left-limits everywhere [discontinuities of the first kind only];
- X is pure-jump (Lévy, Itô) if $\alpha < 2$; BM is $\alpha = 2$ (Wiener).

Stable Processes

Let X_α be a symmetric stable process in $\mathbf{R}^d$ with index $\alpha \in (0\,,2]$; i.e.,

- $E\exp(i\xi\cdot X_\alpha(t)) = \exp(-t|\xi|^\alpha/2)$ for all $\xi \in \mathbf{R}^d$;
- $\{X_\alpha(t+s) - X_\alpha(s)\}_{t\geq 0}$ is independent of $\{X_\alpha(u)\}_{0\leq u\leq s}$;
- $\{X_\alpha(t+s) - X_\alpha(s)\}_{t\geq 0}$ has the same f.d.d.'s are $\{X_\alpha(u)\}_{u\geq 0}$;
- $P\{|X_\alpha(t) - X_\alpha(s)| \geq \epsilon\} \to 0$ as $t \to s$ for all $\epsilon > 0$;
- $t \mapsto X_\alpha(t)$ can be—and will be—assumed to be right-continuous with left-limits everywhere [discontinuities of the first kind only];
- X is pure-jump (Lévy, Itô) if $\alpha < 2$; BM is $\alpha = 2$ (Wiener).
- α has to be in $(0\,,2]$ (Herzog, Bochner, Lévy).

Stable Processes

Let X_α be a symmetric stable process in $\mathbf{R}^d$ with index $\alpha \in (0\,,2]$.

Stable Processes

Let X_α be a symmetric stable process in $\mathbf{R}^d$ with index $\alpha \in (0\,,2]$.

Theorem (McKean, 1955)

For all $A \subset \mathbf{R}^d$ compact,

$$P\left\{X_\alpha([1\,,2]) \cap A \neq \varnothing\right\} > 0 \text{ iff } \mathscr{C}_{d-\alpha}(A) > 0.$$

Stable Processes

Let X_α be a symmetric stable process in $\mathbf{R}^d$ with index $\alpha \in (0\,,2]$.

Theorem (McKean, 1955)

For all $A \subset \mathbf{R}^d$ compact,

$$P\left\{X_\alpha([1\,,2]) \cap A \neq \varnothing\right\} > 0 \textit{ iff } \mathscr{C}_{d-\alpha}(A) > 0.$$

Stable Processes

Let X_α be a symmetric stable process in $\mathbf{R}^d$ with index $\alpha \in (0\,,2]$.

Theorem (McKean, 1955)

For all $A \subset \mathbf{R}^d$ compact,

$$P\left\{X_\alpha([1\,,2]) \cap A \neq \varnothing\right\} > 0 \text{ iff } \mathscr{C}_{d-\alpha}(A) > 0.$$

References:

[Probab] Kakutani (1944), Dvoretzky, Erdős, and Kakutani (1950).
[Analysis] Nevanlinna (1936), Noshiro (1948), Ninomiya (1953).

Stable Processes

McKean: A is hit by X_α iff $\mathscr{C}_{d-\alpha}(A) > 0$

Stable Processes

McKean: A is hit by X_α iff $\mathscr{C}_{d-\alpha}(A) > 0$
Frostman: $d - \dim_{\mathrm{H}} A$ is the critical α such that $\mathscr{C}_{d-\alpha}(A) > 0$.

Stable Processes

McKean: A is hit by X_α iff $\mathscr{C}_{d-\alpha}(A) > 0$
Frostman: $d - \dim_{\rm H} A$ is the critical α such that $\mathscr{C}_{d-\alpha}(A) > 0$.

Corollary (Taylor, 1966)

If $\dim_{\rm H} A \geq d - 2$ *then*

$$\dim_{\rm H} A = \inf\left\{\alpha \in (0\,,2] :\ P\left\{X_\alpha([1\,,2]) \cap A \neq \varnothing\right\} > 0\right\}.$$

Stable Processes

McKean: A is hit by X_α iff $\mathscr{C}_{d-\alpha}(A) > 0$
Frostman: $d - \dim_{\mathrm{H}} A$ is the critical α such that $\mathscr{C}_{d-\alpha}(A) > 0$.

Corollary (Taylor, 1966)

If $\dim_{\mathrm{H}} A \geq d - 2$ *then*

$$\dim_{\mathrm{H}} A = \inf\left\{\alpha \in (0\,,2] : \; P\left\{X_\alpha([1\,,2]) \cap A \neq \varnothing\right\} > 0\right\}.$$

Stable Processes

McKean: A is hit by X_α iff $\mathscr{C}_{d-\alpha}(A) > 0$
Frostman: $d - \dim_{\rm H} A$ is the critical α such that $\mathscr{C}_{d-\alpha}(A) > 0$.

Corollary (Taylor, 1966)

If $\dim_{\rm H} A \geq d - 2$ *then*

$$\dim_{\rm H} A = \inf\{\alpha \in (0\,,2] : \ P\{X_\alpha([1\,,2]) \cap A \neq \varnothing\} > 0\}.$$

What if $\alpha < d - 2$? $\exists$ answer by Peres (1996, 1998) in terms of fractal percolation.

Stable Processes

McKean: A is hit by X_α iff $\mathscr{C}_{d-\alpha}(A) > 0$
Frostman: $d - \dim_{\rm H} A$ is the critical α such that $\mathscr{C}_{d-\alpha}(A) > 0$.

Corollary (Taylor, 1966)

If $\dim_{\rm H} A \geq d - 2$ *then*

$$\dim_{\rm H} A = \inf\left\{\alpha \in (0\,,2] : \ P\left\{X_\alpha([1\,,2]) \cap A \neq \varnothing\right\} > 0\right\}.$$

What if $\alpha < d - 2$? $\exists$ answer by Peres (1996, 1998) in terms of fractal percolation. For processes one can use ideas of Evans (1987a,b), Fitzsimmons and Salisbury (1989), Ren (1990), Hirsch and Song (1994, 1995a-d), Hirsch (1995), ...

Stable Processes

McKean: A is hit by X_α iff $\mathscr{C}_{d-\alpha}(A) > 0$
Frostman: $d - \dim_{\rm H} A$ is the critical α such that $\mathscr{C}_{d-\alpha}(A) > 0$.

Corollary (Taylor, 1966)

If $\dim_{\rm H} A \geq d-2$ *then*

$$\dim_{\rm H} A = \inf\left\{\alpha \in (0\,,2] :\ P\left\{X_\alpha([1\,,2]) \cap A \neq \varnothing\right\} > 0\right\}.$$

What if $\alpha < d-2$? $\exists$ answer by Peres (1996, 1998) in terms of fractal percolation. For processes one can use ideas of Evans (1987a,b), Fitzsimmons and Salisbury (1989), Ren (1990), Hirsch and Song (1994, 1995a-d), Hirsch (1995), ... $\exists$ a simple answer.

Additive Stable Processes

Let $X_1, X_2, \ldots$ be i.i.d. symmetric stable processes in $\mathbf{R}^d$, all with the same stability index $\alpha \in (0\,,2]$, and form the $(N\,,d)$ random field,

$$X_{N,\alpha}(\mathbf{t}) := X_1(t_1) + \cdots + X_N(t_N) \quad \text{for } \mathbf{t} := (t_1\,,\ldots,t_N) \in \mathbf{R}^N_+.$$

Additive Stable Processes

Let $X_1, X_2, \ldots$ be i.i.d. symmetric stable processes in $\mathbf{R}^d$, all with the same stability index $\alpha \in (0\,,2]$, and form the $(N\,,d)$ random field,

$$X_{N,\alpha}(\mathbf{t}) := X_1(t_1) + \cdots + X_N(t_N) \quad \text{for } \mathbf{t} := (t_1\,,\ldots,t_N) \in \mathbf{R}^N_+.$$

Theorem (Hirsch and Song, 1995; Kh. 2002)

Let $A \subset \mathbf{R}^d$ be compact and nonrandom. Then,

$$P\left\{X_{N,\alpha}\left(\mathbf{R}^N_+\right) \cap A \neq \varnothing\right\} > 0 \;\Leftrightarrow\; \mathscr{C}_{d-N\alpha}(A) > 0.$$

A Connection to Harmonic Analysis

If $s \in (0\,, d)$ then the Fourier transform (a la Schwartz) of $|x|^{-s}$ is $c|x|^{s-d}$. "Therefore,"

$$\iint_{\mathbf{R}^d \times \mathbf{R}^d} \frac{\mu(dx)\,\mu(dy)}{|x-y|^s} = c \int_{\mathbf{R}^d} |t|^{s-d}\, |\widehat{\mu}(t)|^2\, dt.$$

A Connection to Harmonic Analysis

If $s \in (0, d)$ then the Fourier transform (a la Schwartz) of $|x|^{-s}$ is $c|x|^{s-d}$. "Therefore,"

$$\iint_{\mathbf{R}^d \times \mathbf{R}^d} \frac{\mu(dx)\,\mu(dy)}{|x-y|^s} = c \int_{\mathbf{R}^d} |t|^{s-d}\,|\widehat{\mu}(t)|^2\,dt.$$

$\Rightarrow$

$$\mathscr{C}_{d-N\alpha}(A) > 0 \quad \Leftrightarrow \quad {}^{\exists}\mu \in \mathscr{P}(A) : \int_{\mathbf{R}^d} \left(\frac{1}{1+|t|^\alpha}\right)^N |\widehat{\mu}(t)|^2\,dt < \infty.$$

A Connection to Harmonic Analysis

If $s \in (0\,,d)$ then the Fourier transform (a la Schwartz) of $|x|^{-s}$ is $c|x|^{s-d}$. "Therefore,"

$$\iint_{\mathbf{R}^d \times \mathbf{R}^d} \frac{\mu(dx)\,\mu(dy)}{|x-y|^s} = c \int_{\mathbf{R}^d} |t|^{s-d}\, |\widehat{\mu}(t)|^2 \, dt.$$

$\Rightarrow$

$$\mathscr{C}_{d-N\alpha}(A) > 0 \quad \Leftrightarrow \quad {}^{\exists}\mu \in \mathscr{P}(A) : \int_{\mathbf{R}^d} \left(\frac{1}{1+|t|^{\alpha}} \right)^N |\widehat{\mu}(t)|^2 \, dt < \infty.$$

This can be improved generically.

A More Generic Variation

Additive Lévy Processes

$X_1, X_2, \ldots$ Lévy processes in $\mathbf{R}^d$;

A More Generic Variation

Additive Lévy Processes

$X_1, X_2, \ldots$ Lévy processes in $\mathbf{R}^d$;
$E \exp(i\xi \cdot X_j(t)) = \exp(-t\Psi_j(\xi))$;

A More Generic Variation

Additive Lévy Processes

$X_1, X_2, \ldots$ Lévy processes in $\mathbf{R}^d$;
$E \exp(i\xi \cdot X_j(t)) = \exp(-t\Psi_j(\xi))$;
$\Psi_1, \ldots, \Psi_N : \mathbf{R}^d \to \mathbf{C}$, “neg. def.” (Schoenberg)

A More Generic Variation

Additive Lévy Processes

$X_1, X_2, \ldots$ Lévy processes in $\mathbf{R}^d$;
$E \exp(i\xi \cdot X_j(t)) = \exp(-t\Psi_j(\xi))$;
$\Psi_1, \ldots, \Psi_N : \mathbf{R}^d \to \mathbf{C}$, "neg. def." (Schoenberg)

$$X(\mathbf{t}) := X_1(t_1) + \cdots + X_N(t_N) \quad \text{"ALP"}$$

A More Generic Variation

Additive Lévy Processes

Theorem (Kh. and Xiao, 2006)

$X(\mathbf{R}_+^N) \oplus A$ can have positive Leb. meas. iff $\exists \mu \in \mathscr{P}(A)$ such that

$$\int_{\mathbf{R}^d} \prod_{j=1}^{N} \mathrm{Re}\left(\frac{1}{1+\Psi_j(t)}\right) |\widehat{\mu}(t)|^2 \, dt < \infty.$$

A More Generic Variation

Additive Lévy Processes

Theorem (Kh. and Xiao, 2006)

$X(\mathbf{R}_+^N) \oplus A$ can have positive Leb. meas. iff $\exists \mu \in \mathscr{P}(A)$ such that

$$\int_{\mathbf{R}^d} \prod_{j=1}^{N} \mathrm{Re}\left(\frac{1}{1+\Psi_j(t)} \right) |\widehat{\mu}(t)|^2 \, dt < \infty.$$

- Improves older versions (Orey, 1967; Kesten, 1969; Hawkes, 1984; Kh., Xiao, and Zhong, 2003).

A More Generic Variation

Additive Lévy Processes

Theorem (Kh. and Xiao, 2006)

$X(\mathbf{R}_+^N) \oplus A$ can have positive Leb. meas. iff $\exists \mu \in \mathscr{P}(A)$ such that

$$\int_{\mathbf{R}^d} \prod_{j=1}^{N} \mathrm{Re}\left(\frac{1}{1+\Psi_j(t)}\right) |\widehat{\mu}(t)|^2 \, dt < \infty.$$

- Improves older versions (Orey, 1967; Kesten, 1969; Hawkes, 1984; Kh., Xiao, and Zhong, 2003).
- Proof is very long.

A More Generic Variation

Additive Lévy Processes

Let X be an ALP $(\Psi_1, \ldots, \Psi_N)$ and Y an indept M-parameter add. stable α. Then:

- $X(\mathbf{R}_+^N) \oplus Y(\mathbf{R}_+^M)$ is the range of the $(N+M, d)$ ALP

$$Z(\mathbf{t} \otimes \mathbf{s}) := X(\mathbf{t}) + Y(\mathbf{s}) \qquad \text{for } \mathbf{t} \in \mathbf{R}_+^N, \mathbf{s} \in \mathbf{R}_+^M.$$

A More Generic Variation

Additive Lévy Processes

Let X be an ALP $(\Psi_1, \ldots, \Psi_N)$ and Y an indept M-parameter add. stable α. Then:

- $X(\mathbf{R}_+^N) \oplus Y(\mathbf{R}_+^M)$ is the range of the $(N+M, d)$ ALP

$$Z(\mathbf{t} \otimes \mathbf{s}) := X(\mathbf{t}) + Y(\mathbf{s}) \qquad \text{for } \mathbf{t} \in \mathbf{R}_+^N, \mathbf{s} \in \mathbf{R}_+^M.$$

- "Conditional on $X(\mathbf{R}_+^M) = A$," $X(\mathbf{R}_+^N) \oplus Y(\mathbf{R}_+^M) = A \oplus Y(\mathbf{R}_+^N)$.

A More Generic Variation

Additive Lévy Processes

Let X be an ALP $(\Psi_1, \ldots, \Psi_N)$ and Y an indept M-parameter add. stable α. Then:

- $X(\mathbf{R}_+^N) \oplus Y(\mathbf{R}_+^M)$ is the range of the $(N+M, d)$ ALP

$$Z(\mathbf{t} \otimes \mathbf{s}) := X(\mathbf{t}) + Y(\mathbf{s}) \qquad \text{for } \mathbf{t} \in \mathbf{R}_+^N, \mathbf{s} \in \mathbf{R}_+^M.$$

- "Conditional on $X(\mathbf{R}_+^M) = A$," $X(\mathbf{R}_+^N) \oplus Y(\mathbf{R}_+^M) = A \oplus Y(\mathbf{R}_+^N)$.

A More Generic Variation

Additive Lévy Processes

Let X be an ALP $(\Psi_1, \ldots, \Psi_N)$ and Y an indept M-parameter add. stable α. Then:

- $X(\mathbf{R}_+^N) \oplus Y(\mathbf{R}_+^M)$ is the range of the $(N+M, d)$ ALP

$$Z(\mathbf{t} \otimes \mathbf{s}) := X(\mathbf{t}) + Y(\mathbf{s}) \qquad \text{for } \mathbf{t} \in \mathbf{R}_+^N, \mathbf{s} \in \mathbf{R}_+^M.$$

- "Conditional on $X(\mathbf{R}_+^M) = A$," $X(\mathbf{R}_+^N) \oplus Y(\mathbf{R}_+^M) = A \oplus Y(\mathbf{R}_+^N)$.

Corollary (Kh. and Xiao, 2006)

$$\dim_{\mathrm{H}} X(\mathbf{R}_+^N) = \sup\left\{ s > 0 : \int_{\mathbf{R}^d} \prod_{j=1}^{N} \mathrm{Re}\left(\frac{1}{1+\Psi_j(t)} \right) \frac{dt}{|t|^{d-s}} < \infty \right\}.$$

A More Generic Variation

Additive Lévy Processes

References: Taylor (1952/53), McKean (1959), Blumenthal and Getoor (1960, 1961), Pruitt (1969), Fristedt (1974), Kh., Xiao, and Zhong (2003), Kh. and Xiao (2006).

The One-Parameter Case

Let X be a Lévy process in $\mathbf{R}^d$. Then (Kh., Xiao, and Zhong, 2003)

$$\dim_{\mathrm{H}} X(\mathbf{R}_+) = \sup\left\{ s > 0 : \int_{\mathbf{R}^d} \mathrm{Re}\left(\frac{1}{1+\Psi(t)}\right) \frac{dt}{|t|^{d-s}} < \infty \right\}.$$

"Hendricks's index"

The One-Parameter Case

Let X be a Lévy process in $\mathbf{R}^d$. Then (Kh., Xiao, and Zhong, 2003)

$$\dim_{\mathrm{H}} X(\mathbf{R}_+) = \sup\left\{ s > 0 : \int_{\mathbf{R}^d} \mathrm{Re}\left(\frac{1}{1+\Psi(t)}\right) \frac{dt}{|t|^{d-s}} < \infty \right\}.$$

“Hendricks’s index”

Also,

$$\dim_{\mathrm{H}} X(\mathbf{R}_+) = \sup\left\{ \alpha > 0 : \liminf_{r \to 0} \frac{1}{r^\alpha} \int_0^\infty P\{|X(s)| \le r\} e^{-s}\, ds = 0 \right\}.$$

“Pruitt’s index”

The One-Parameter Case

Define

$$W(r) := \int_{\mathbf{R}^d} \frac{\kappa(x/r)}{\prod_{j=1}^{d}(1+x_j^2)}\,dx \quad \text{where} \quad \kappa(t) := \mathrm{Re}\left(\frac{1}{1+\Psi(t)}\right).$$

Theorem (Kh. and Xiao, 2006)

$$\dim_{\mathrm{H}} X(\mathbf{R}_+) = \liminf_{r \downarrow 0} \frac{\log W(r)}{\log r}.$$

The One-Parameter Case

Define

$$W(r) := \int_{\mathbf{R}^d} \frac{\kappa(x/r)}{\prod_{j=1}^d (1+x_j^2)}\,dx \quad \text{where} \quad \kappa(t) := \mathrm{Re}\left(\frac{1}{1+\Psi(t)}\right).$$

Theorem (Kh. and Xiao, 2006)

$$\dim_{\mathrm{H}} X(\mathbf{R}_+) = \liminf_{r \downarrow 0} \frac{\log W(r)}{\log r}.$$

The One-Parameter Case

Define

$$W(r) := \int_{\mathbf{R}^d} \frac{\kappa(x/r)}{\prod_{j=1}^d (1 + x_j^2)} \, dx \quad \text{where} \quad \kappa(t) := \mathrm{Re}\left(\frac{1}{1 + \Psi(t)}\right).$$

Theorem (Kh. and Xiao, 2006)

$$\dim_{\mathrm{H}} X(\mathbf{R}_+) = \liminf_{r \downarrow 0} \frac{\log W(r)}{\log r}.$$

$\limsup = \dim_{\mathrm{P}} X(\mathbf{R}_+)$ (Kh. Xiao, 2006)

The One-Parameter Case

Define

$$W(r) := \int_{\mathbf{R}^d} \frac{\kappa(x/r)}{\prod_{j=1}^d (1+x_j^2)}\, dx \quad \text{where} \quad \kappa(t) := \mathrm{Re}\left(\frac{1}{1+\Psi(t)}\right).$$

Theorem (Kh. and Xiao, 2006)

$$\dim_{\mathrm{H}} X(\mathbf{R}_+) = \liminf_{r \downarrow 0} \frac{\log W(r)}{\log r}.$$

$\limsup = \dim_{\mathrm{P}} X(\mathbf{R}_+)$ (Kh. Xiao, 2006)

Open Problem: What if X is ALP?

Derivation of the Dimension Formula

F.T.: $(\mathscr{F}f)(z) = \int_{\mathbf{R}^d} e^{iz\cdot x} f(x)\, dx$

Derivation of the Dimension Formula

F.T.: $(\mathscr{F}f)(z) = \int_{\mathbf{R}^d} e^{iz\cdot x} f(x)\, dx$

$$\phi_r(x) := \prod_{j=1}^{d} \frac{1-\cos(2rx_j)}{2\pi r x_j^2}.$$

- $\phi_r \geq 0$, $\int_{\mathbf{R}^d} \phi_r(x)\, dx < \infty$

Derivation of the Dimension Formula

F.T.: $(\mathscr{F}f)(z) = \int_{\mathbf{R}^d} e^{iz\cdot x} f(x)\,dx$

$$\phi_r(x) := \prod_{j=1}^{d} \frac{1-\cos(2rx_j)}{2\pi r x_j^2}.$$

- $\phi_r \geq 0$, $\int_{\mathbf{R}^d} \phi_r(x)\,dx < \infty$
- $(\mathscr{F}\phi_r)(z) = \prod_{j=1}^{d}(1-|z_j|/(2r))^+$

Derivation of the Dimension Formula

F.T.: $(\mathscr{F}f)(z) = \int_{\mathbf{R}^d} e^{iz\cdot x} f(x)\,dx$

$$\phi_r(x) := \prod_{j=1}^{d} \frac{1-\cos(2rx_j)}{2\pi r x_j^2}.$$

- $\phi_r \geq 0$, $\int_{\mathbf{R}^d} \phi_r(x)\,dx < \infty$
- $(\mathscr{F}\phi_r)(z) = \prod_{j=1}^{d}(1-|z_j|/(2r))^+$
- $0 \leq \mathscr{F}\phi_r \leq 1$

Derivation of the Dimension Formula

F.T.: $(\mathscr{F}f)(z) = \int_{\mathbf{R}^d} e^{iz\cdot x} f(x)\,dx$

$$\phi_r(x) := \prod_{j=1}^{d} \frac{1-\cos(2rx_j)}{2\pi r x_j^2}.$$

- $\phi_r \geq 0$, $\int_{\mathbf{R}^d} \phi_r(x)\,dx < \infty$
- $(\mathscr{F}\phi_r)(z) = \prod_{j=1}^{d}(1-|z_j|/(2r)))^+$
- $0 \leq \mathscr{F}\phi_r \leq 1$
- $z \in B(0,r) \Rightarrow 1-(2r)^{-1}|z_j| \geq \frac{1}{2}$

Derivation of the Dimension Formula

F.T.: $(\mathscr{F}f)(z) = \int_{\mathbf{R}^d} e^{iz\cdot x} f(x)\, dx$

$$\phi_r(x) := \prod_{j=1}^{d} \frac{1-\cos(2rx_j)}{2\pi r x_j^2}.$$

- $\phi_r \geq 0$, $\int_{\mathbf{R}^d} \phi_r(x)\, dx < \infty$
- $(\mathscr{F}\phi_r)(z) = \prod_{j=1}^{d} (1 - |z_j|/(2r)))^+$
- $0 \leq \mathscr{F}\phi_r \leq 1$
- $z \in B(0\,, r) \Rightarrow 1 - (2r)^{-1}|z_j| \geq \frac{1}{2}$

Derivation of the Dimension Formula

F.T.: $(\mathscr{F}f)(z) = \int_{\mathbf{R}^d} e^{iz\cdot x} f(x)\,dx$

$$\phi_r(x) := \prod_{j=1}^{d} \frac{1-\cos(2rx_j)}{2\pi r x_j^2}.$$

- $\phi_r \geq 0$, $\int_{\mathbf{R}^d} \phi_r(x)\,dx < \infty$
- $(\mathscr{F}\phi_r)(z) = \prod_{j=1}^{d}(1 - |z_j|/(2r)))^+$
- $0 \leq \mathscr{F}\phi_r \leq 1$
- $z \in B(0\,,r) \Rightarrow 1-(2r)^{-1}|z_j| \geq \frac{1}{2} \Rightarrow \mathbf{1}_{B(0,r)}(z) \leq 2^d\,(\mathscr{F}\phi_r)(z)$

$\Rightarrow$

$$P\{|X(s)| \le r\} \le 2^d E\left[(\mathscr{F}\phi_r)(X(s))\right]$$

Derivation of the Dimension Formula

Upper Bound

$\Rightarrow$

$$P\{|X(s)| \le r\} \le 2^d E\left[(\mathscr{F}\phi_r)(X(s))\right] = 2^d \int \phi_r(\xi) e^{-s\Psi(\xi)}\, d\xi$$

Derivation of the Dimension Formula

Upper Bound

$\Rightarrow$

$$P\{|X(s)| \le r\} \le 2^d E\left[(\mathscr{F}\phi_r)(X(s))\right] = 2^d \int \phi_r(\xi) e^{-s\Psi(\xi)}\, d\xi$$

$\Rightarrow$

$$\int_0^\infty P\{|X(s)| \le r\} e^{-s}\, ds \quad \le \quad 2^d \int_{\mathbf{R}^d} \kappa(x)\phi_r(x)\, dx$$

Derivation of the Dimension Formula

Upper Bound

$\Rightarrow$

$$P\{|X(s)| \le r\} \le 2^d E\left[(\mathscr{F}\phi_r)(X(s))\right] = 2^d \int \phi_r(\xi) e^{-s\Psi(\xi)}\, d\xi$$

$\Rightarrow$

$$\begin{aligned} \int_0^\infty P\{|X(s)| \le r\} e^{-s}\, ds &\le 2^d \int_{\mathbf{R}^d} \kappa(x)\phi_r(x)\, dx \\ &\le 2^d W(r), \end{aligned}$$

because $(1 - \cos z)/z^2 \le 2\pi/(1+z^2)$.

Derivation of the Dimension Formula

Lower Bound

- $S(t) := (S_1(t), \ldots, S_d(t))$

Derivation of the Dimension Formula

Lower Bound

- $S(t) := (S_1(t), \ldots, S_d(t))$
- $S_1, \ldots, S_d$ indept of each other and X

Derivation of the Dimension Formula

Lower Bound

- $S(t) := (S_1(t), \ldots, S_d(t))$
- $S_1, \ldots, S_d$ indept of each other and X
- Cauchy processes in **R**, all with the same characteristic function $E[e^{izS_1(t)}] = e^{-t|z|}$.

Derivation of the Dimension Formula

Lower Bound

$$E\left[\exp\{iX(t)\cdot S(\lambda)\}\right] = E\left[\exp\{-\lambda \textstyle\sum_{j=1}^{d} |X_j(t)|\}\right]$$

Derivation of the Dimension Formula

Lower Bound

$$E[\exp\{iX(t)\cdot S(\lambda)\}] = E[\exp\{-\lambda \textstyle\sum_{j=1}^d |X_j(t)|\}]$$

$$E\left[e^{iX(t)\cdot S(\lambda)}\right] = E\left[e^{-t\Psi(S(\lambda))}\right] = \frac{1}{\pi^d}\int_{\mathbf{R}^d} \frac{e^{-t\Psi(\lambda x)}}{\prod_{j=1}^d (1+x_j^2)}\,dx. \qquad (1)$$

Derivation of the Dimension Formula

Lower Bound

$E[\exp\{iX(t)\cdot S(\lambda)\}] = E[\exp\{-\lambda\sum_{j=1}^{d}|X_j(t)|\}]$

$$E\left[e^{iX(t)\cdot S(\lambda)}\right] = E\left[e^{-t\Psi(S(\lambda))}\right] = \frac{1}{\pi^d}\int_{\mathbf{R}^d}\frac{e^{-t\Psi(\lambda x)}}{\prod_{j=1}^{d}(1+x_j^2)}\,dx. \qquad (1)$$

$\Rightarrow \exp\{-(k/r)\sum_{j=1}^{d}|x_j|\} \le \mathbf{1}_{B(0,r)}(x) + e^{-k}\mathbf{1}_{B(0,r)^c}(x).$

Derivation of the Dimension Formula

Lower Bound

$E[\exp\{iX(t)\cdot S(\lambda)\}] = E[\exp\{-\lambda \sum_{j=1}^{d} |X_j(t)|\}]$

$$E\left[e^{iX(t)\cdot S(\lambda)}\right] = E\left[e^{-t\Psi(S(\lambda))}\right] = \frac{1}{\pi^d}\int_{\mathbf{R}^d} \frac{e^{-t\Psi(\lambda x)}}{\prod_{j=1}^{d}(1+x_j^2)}\,dx. \qquad (1)$$

$\Rightarrow \exp\{-(k/r)\sum_{j=1}^{d}|x_j|\} \le \mathbf{1}_{B(0,r)}(x) + e^{-k}\mathbf{1}_{B(0,r)^c}(x).$

$\Rightarrow \mathbf{1}_{B(0,r)}(x) \ge E[\exp\{ix\cdot S(k/r)\}] - e^{-k}\mathbf{1}_{B(0,r)^c}(x)$

Derivation of the Dimension Formula

Lower Bound

$$E[\exp\{iX(t)\cdot S(\lambda)\}] = E[\exp\{-\lambda \textstyle\sum_{j=1}^{d} |X_j(t)|\}]$$

$$E\left[e^{iX(t)\cdot S(\lambda)}\right] = E\left[e^{-t\Psi(S(\lambda))}\right] = \frac{1}{\pi^d}\int_{\mathbf{R}^d} \frac{e^{-t\Psi(\lambda x)}}{\prod_{j=1}^{d}(1+x_j^2)}\,dx. \qquad (1)$$

$\Rightarrow \exp\{-(k/r)\sum_{j=1}^{d}|x_j|\} \le \mathbf{1}_{B(0,r)}(x) + e^{-k}\mathbf{1}_{B(0,r)^c}(x).$

$\Rightarrow \mathbf{1}_{B(0,r)}(x) \ge E[\exp\{ix\cdot S(k/r)\}] - e^{-k}\mathbf{1}_{B(0,r)^c}(x)$

$\Rightarrow (1-e^{-k})\mathbf{1}_{B(0,r)}(x) \ge E[\exp\{ix\cdot S(k/r)\}] - e^{-k}$

$E[\exp\{iX(t)\cdot S(\lambda)\}] = E[\exp\{-\lambda\sum_{j=1}^{d}|X_j(t)|\}]$

$$E\left[e^{iX(t)\cdot S(\lambda)}\right] = E\left[e^{-t\Psi(S(\lambda))}\right] = \frac{1}{\pi^d}\int_{\mathbf{R}^d}\frac{e^{-t\Psi(\lambda x)}}{\prod_{j=1}^{d}(1+x_j^2)}\,dx. \qquad (1)$$

$\Rightarrow \exp\{-(k/r)\sum_{j=1}^{d}|x_j|\} \le \mathbf{1}_{B(0,r)}(x) + e^{-k}\mathbf{1}_{B(0,r)^c}(x).$

$\Rightarrow \mathbf{1}_{B(0,r)}(x) \ge E[\exp\{ix\cdot S(k/r)\}] - e^{-k}\mathbf{1}_{B(0,r)^c}(x)$

$\Rightarrow (1-e^{-k})\mathbf{1}_{B(0,r)}(x) \ge E[\exp\{ix\cdot S(k/r)\}] - e^{-k}$

$\Rightarrow$

$$(1-e^{-k})\int_0^\infty P\{|X(s)|\le r\}e^{-s}\,ds \ge -e^{-k} + \frac{1}{\pi^d}W(r/k).$$

Derivation of the Dimension Formula

Lower Bound

$E[\exp\{iX(t)\cdot S(\lambda)\}] = E[\exp\{-\lambda \sum_{j=1}^{d} |X_j(t)|\}]$

$$E\left[e^{iX(t)\cdot S(\lambda)}\right] = E\left[e^{-t\Psi(S(\lambda))}\right] = \frac{1}{\pi^d}\int_{\mathbf{R}^d} \frac{e^{-t\Psi(\lambda x)}}{\prod_{j=1}^{d}(1+x_j^2)}\,dx. \qquad (1)$$

$\Rightarrow \exp\{-(k/r)\sum_{j=1}^{d}|x_j|\} \leq \mathbf{1}_{B(0,r)}(x) + e^{-k}\mathbf{1}_{B(0,r)^c}(x)$.

$\Rightarrow \mathbf{1}_{B(0,r)}(x) \geq E[\exp\{ix\cdot S(k/r)\}] - e^{-k}\mathbf{1}_{B(0,r)^c}(x)$

$\Rightarrow (1-e^{-k})\mathbf{1}_{B(0,r)}(x) \geq E[\exp\{ix\cdot S(k/r)\}] - e^{-k}$

$\Rightarrow$

$$(1-e^{-k})\int_0^\infty P\{|X(s)|\leq r\}e^{-s}\,ds \geq -e^{-k} + \frac{1}{\pi^d}W(r/k).$$

Set $k := r^{-\varepsilon}$ to finish.