Tutorial on Additive Lévy Processes

Lecture #1

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Some Terminology

- An “(N, d) random field” X has N parameters and takes values in $\mathbf{R}^d$ (Adler, 1981);
  i.e., $X(\mathbf{t}) \in \mathbf{R}^d$ for all $\mathbf{t} := (t_1, \ldots, t_N) \in \mathbf{R}^N$.

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- $X_1,\ldots,X_N$ independent Brownian motions in $\mathbf{R}^d$.
  "Additive Brownian motion":

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- Likewise, can have "additive stable," "additive Lévy," etc.

ABM and the Local Dynamics of Brownian Sheet

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Lemma

If $A_1, A_2, \ldots$ are nonrandom and disjoint then a.s.,

$$\dot{W}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \dot{W}(A_n),$$

as long as $\mathrm{meas}(A_n) < \infty$ for all n.

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- $(N, 1)$ "Brownian sheet" $B :=$ the dF of $\dot{W}$; i.e.,

$$B(\mathbf{t}) := \dot{W}\left([0, t_1] \times \cdots \times [0, t_N]\right) \quad \text{for all } \mathbf{t} = (t_1, \ldots, t_N) \in \mathbf{R}_+^N.$$

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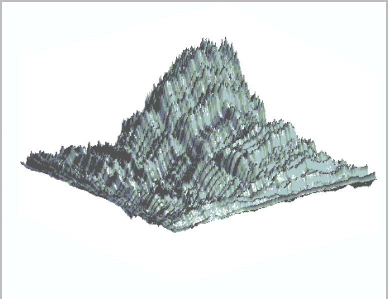
- In many cases, "SPDEs look like B." [Girsanov]
- $(N\,,d)$ Brownian sheet $B(\mathbf{t}) := (B_1(\mathbf{t})\,,\ldots,B_d(\mathbf{t}))$, where $B_1,\ldots,B_d$ are indept. $(N\,,1)$ Brownian sheets.

ABM and the Local Dynamics of Brownian Sheet



$(2\,,1)$ Brownian sheet

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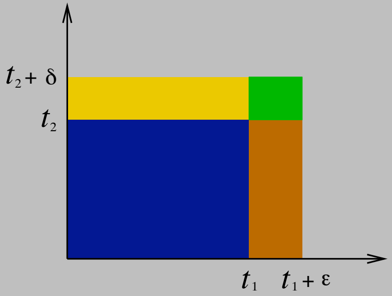

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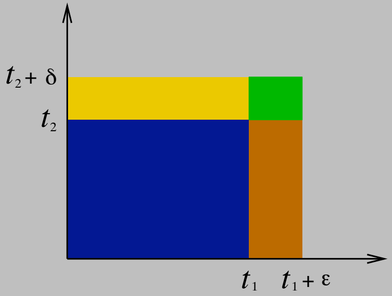

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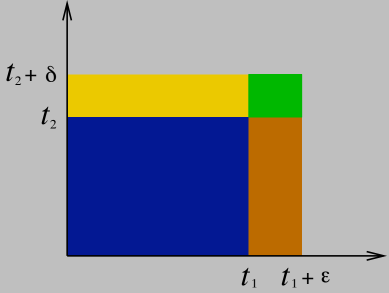

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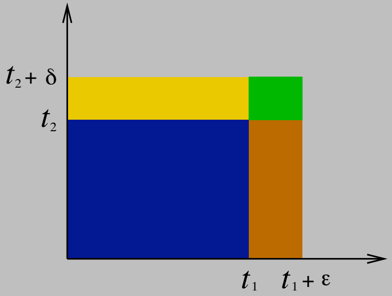

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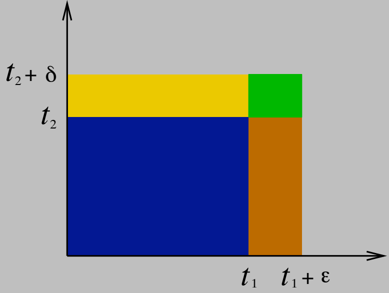

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- $Z(\epsilon\,,\delta) := \dot W({\color{green}\blacksquare}) =$ BS

ABM and the Local Dynamics of Brownian Sheet

$$B(1+\epsilon, 1+\delta) - \overbrace{B(1,1)}^{\dot{W}({\color{blue}\blacksquare})} = \overbrace{X(\epsilon)+Y(\delta)}^{\dot{W}({\color{brown}\blacksquare})+\dot{W}({\color{yellow}\blacksquare})=\text{ABM}} + \overbrace{Z(\epsilon,\delta)}^{\dot{W}({\color{green}\blacksquare})=\text{BS}}$$

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References: Orey and Pruitt (1973), Kendall (1980), Ehm (1981), Dalang and Walsh (1992; 1993; 1996), Kh. (1995; 1999; 2003), Dalang and Mountford (1996; 1997; 2001), Kh. and Shi (1999), Kh. Xiao (2005), Kh. Xiao, and Wu (2006) . . . .

First Application: Contours of Brownian Sheet

Kendall's Theorem

Let $B = (2\,,1)$ BS; choose and fix $s, t > 0$.

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Theorem (Kendall, 1980)

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- WLOG $s = t = 1$.

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Kendall's Theorem, Continued

- $C(r) := \{(x, y) \in \mathbf{R}^2 : |x - 1| \vee |y - 1| \leq r\}$.

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- I will sketch the proof of a slightly weaker variant. [In fact it is equivalent.]

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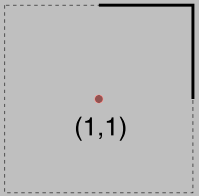

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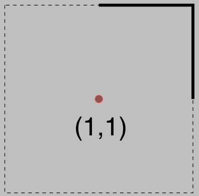

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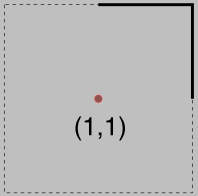

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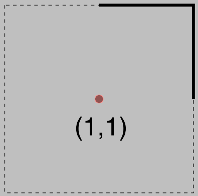

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- Goal: $\lim_{r \downarrow 0} P(J'(r)) > 0$.
- Local dynamics + scaling $\Rightarrow$

$$P(J'(r)) \to P\left\{X(1) + Y(1) > \sup_{(u,v) \in C'(1)} (X(u) + Y(v))\right\} > 0.$$

The Zero-Set of *B*

Kendall's theorem holds for "most" points in the zero-set too.
Can you see it?

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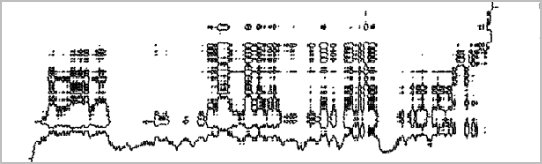

(Kh., Révész, and Shi, 2005)

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem

X and Y = independent BMs in $\mathbf{R}^d$. Then:
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Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

Equivalently, TFAE:

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Key idea: $X([1,2]) \cap Y([1,2]) \neq \varnothing \Leftrightarrow$

$$\overbrace{X(s) - Y(t)}^{A(s,t)=\text{ABM}} = 0 \text{ for some } (s,t) \in [1,2]^2,$$

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$\Leftrightarrow$

$$0 \in A\left([1,2]^2\right).$$

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

- When $d \le 3$ one can directly construct a "local time." This is an a.s.-nontrivial random measure on the set $A^{-1}\{0\} = \{(s\,,t) \in [1\,,2]^2 : X(s) = Y(t)\}$. Therefore, $0 \in A([1\,,2]^2)$ a.s.

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

- When $d \le 3$ one can directly construct a "local time." This is an a.s.-nontrivial random measure on the set $A^{-1}\{0\} = \{(s\,,t) \in [1\,,2]^2 : X(s) = Y(t)\}$. Therefore, $0 \in A([1\,,2]^2)$ a.s.
- We prove that if $d \ge 5$ then $A(s\,,t) = X(s) - Y(t) \neq 0$ for all $s,t \in [1\,,2]$. This proof can be pushed through when $d = 4$ but needs more care; see the original paper (Kh., *Expos. Math.*, 2003).

Second Application: Intersections of Brownian Motions

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Exercise

Prove that $A([1,2]^2) = A(1,1) + \bar{A}([0,1]^2)$, where $\bar{A}$ is a copy of A, independent of $A(1,1)$, and $a + S = \{a + s : s \in S\}$ for all $a \in \mathbf{R}^d$ and $S \subset \mathbf{R}^d$.

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

Let $A(s,t) := X(s) - Y(t)$, $d \geq 5$; $m :=$ Leb. meas. on $\mathbf{R}^d$.
By the Exercise,

$$P\left\{0 \in A\left([1,2]^2\right)\right\}$$

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

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Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

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Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

Let $A(s,t) := X(s) - Y(t)$, $d \geq 5$; $m :=$ Leb. meas. on $\mathbf{R}^d$.
By the Exercise,

$$
\begin{aligned}
P\left\{0 \in A\left([1,2]^2\right)\right\} &= \int_{\mathbf{R}^d} P\left\{x \in A\left([0,1]^2\right)\right\} \underbrace{P\left\{A(1,1) \in -dx\right\}}_{=\varphi(x)dx,\ \varphi>0} \\
&\asymp \int_{\mathbf{R}^d} P\left\{x \in A\left([0,1]^2\right)\right\} dx
\end{aligned}
$$

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

Let $A(s\,,t) := X(s) - Y(t)$, $d \geq 5$; $m :=$ Leb. meas. on $\mathbf{R}^d$.
By the Exercise,

$$\begin{aligned} P\left\{0 \in A\left([1\,,2]^2\right)\right\} &= \int_{\mathbf{R}^d} P\left\{x \in A\left([0\,,1]^2\right)\right\} \underbrace{P\left\{A(1\,,1) \in -dx\right\}}_{=\varphi(x)dx,\ \varphi>0} \\ &\asymp \int_{\mathbf{R}^d} P\left\{x \in A\left([0\,,1]^2\right)\right\} dx \\ &= E\left\{m\left(A\left([0\,,1]^2\right)\right)\right\}. \end{aligned}$$

(Lévy, 1940; Kahane, 1983) "$a \asymp b$" means "$a > 0 \Leftrightarrow b > 0$."

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

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Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

Let $A(s\,,t) := X(s) - Y(t)$, $d \geq 5$; $m :=$ Leb. meas. on $\mathbf{R}^d$.
Goal: $Em(A([0,1]^2)) = 0$.

Second Application: Intersections of Brownian Motions

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By Brownian scaling,

$$\underbrace{A([0\,,2]^2)}$$

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By Brownian scaling,

$$\underbrace{A([0,2]^2)}_{\sim X(2s)-Y(2t)}$$

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Goal: $Em(A([0,1]^2)) = 0$.

By Brownian scaling,

$$\underbrace{A([0\,,2]^2)}_{\sim X(2s)-Y(2t)} \overset{\mathscr{D}}{=} \sqrt{2}A([0\,,1]^2)$$

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

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Goal: $Em(A([0,1]^2)) = 0$.

By Brownian scaling,

$$\underbrace{A([0\,,2]^2)}_{\sim X(2s)-Y(2t)} \stackrel{\mathscr{D}}{=} \sqrt{2}A([0\,,1]^2)$$

$$\Leftrightarrow\ m(A([0\,,2]^2) \stackrel{\mathscr{D}}{=} 2^{d/2}m(A[0\,,1]^2)$$

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

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Second Application: Intersections of Brownian Motions

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Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

Let $A(s\,,t) := X(s) - Y(t)$, $d \geq 5$; $m :=$ Leb. meas. on $\mathbf{R}^d$.
We know: $Em(A([0\,,2]^2) = 2^{d/2}Em(A([0\,,1]^2)$.

Second Application: Intersections of Brownian Motions

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Second Application: Intersections of Brownian Motions

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We know: $Em(A([0,2]^2) = 2^{d/2}Em(A([0,1]^2)$.
We want: $Em(A([0,1]^2) = 0$.
Note that

$$\begin{aligned} A\left([0,2]^2\right) &= A\left([0,1]^2\right) \cup A\left([0,1] \times [1,2]\right) \\ &\quad \cup A\left([1,2] \times [0,1]\right) \cup A\left([1,2]^2\right). \end{aligned}$$

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

Let $A(s,t) := X(s) - Y(t)$, $d \geq 5$; $m :=$ Leb. meas. on $\mathbf{R}^d$.
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$$A\left([0,2]^2\right) = A\left([0,1]^2\right) \cup A([0,1] \times [1,2]) \\ \cup A([1,2] \times [0,1]) \cup A\left([1,2]^2\right).$$

Therefore,

$$m\left(A\left([0,2]^2\right)\right) \leq m\left(A\left([0,1]^2\right)\right) + m(A([0,1] \times [1,2])) \\ + m(A([1,2] \times [0,1])) + m\left(A\left([1,2]^2\right)\right).$$

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

We know:

$$\begin{aligned} m\left(A\left([0\,,2]^2\right)\right) &\le m\left(A\left([0\,,1]^2\right)\right) + m\left(A\left([0\,,1]\times[1\,,2]\right)\right) \\ &\quad + m\left(A\left([1\,,2]\times[0\,,1]\right)\right) + m\left(A\left([1\,,2]^2\right)\right). \end{aligned}$$

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

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By the Exercise,

$$\Rightarrow\; m\left(A\left([1\,,2]^2\right)\right) \stackrel{\mathscr{D}}{=} m\left(A\left([0\,,1]^2\right)\right).$$

Same with the other terms in the first display.

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

Thus,

$$Em\left(A\left([0\,,2]^2\right)\right) \le 4Em\left(A\left([0\,,1]^2\right)\right)$$

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

Thus,

$$Em\left(A\left([0\,,2]^2\right)\right) \le 4Em\left(A\left([0\,,1]^2\right)\right)$$

$\Rightarrow$

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Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

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Second Application: Intersections of Brownian Motions

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$d \ge 5 \Rightarrow Em(A([0\,,1]^2)) = 0$, as desired.

Second Application: Intersections of Brownian Motions

The Dvoretzky–Erdős–Kakutani Theorem, Continued

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$d \ge 5 \Rightarrow Em(A([0\,,1]^2)) = 0$, as desired.

Exercise

Prove that $Em(A([0\,,1]^2) < \infty$.

In Memory of Ron Pyke (1931–2005)