On Some Applications of Stable Processes

Davar Khoshnevisan

Department of Mathematics
University of Utah
http://www.math.utah.edu/~davar

The Third Erich L. Lehmann Symposium
May 16–19, 2007

Objectives

- A statistical idea in anamolous “diffusions”

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- Stable processes in analysis

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- $g_n := y(s_1) + \cdots + y(s_n)$ = random walk in random scenery (Kesten and Spitzer, 1979)

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An iterated logarithm law

Theorem (Kh–Lewis, 1998)

$\exists c \in (0\,,\infty)$ *s.t.*

$$\limsup_{t\to\infty}\left(\frac{\ln\ln t}{t}\right)^{1-1/(2\alpha)}\frac{G(t)}{(\ln\ln t)^{3/2}} = c \qquad \text{a.s.}$$

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- Interest here: A Borel–Cantelli lemma

Positive quadrant dependence

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- $\sum_{k=1}^{\infty} \mathrm{P}\{Z_k \geq 0\} = \infty$; *and*
- *there exists an integer sequence $n_1, n_2, \ldots \nearrow \infty$ s.t.*

$$\sum_{1\leq j<k\leq n_\ell} \mathrm{Cov}(Z_j\,, Z_k) = o\left(\left|\sum_{k=1}^{n_\ell} \mathrm{P}\{Z_k \geq 0\}\right|^2\right)$$

Hausdorff measure

- Given a set $A \subset \mathbf{R}^m$ and $\epsilon, s > 0$,

$$\mathscr{H}_s^\epsilon(A) := \inf\left\{ \sum_{i=1}^\infty |A_i|^s : \; A \subseteq \bigcup_{j=1}^\infty A_j, \; \sup_{k\geq 1} |A_k| \leq \epsilon \right\}.$$

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- If $s = m$, then that restriction *is* Lebesgue's measure, provided that we choose the diameter "$|\cdots|$" correctly.

Hausdorff dimension

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- $\dim_{\mathrm{H}} A$ = the "Hausdorff dimension" of A.

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- $m - 2 \leq m - \alpha < m$ (!)

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- Let $Y(t) := X(t) - X'(t)$, where X' is an independent copy of X [symmetrization]. Then, $\dim_{\mathrm{H}} Y(\mathbf{R}_+) \le \dim_{\mathrm{H}} X(\mathbf{R}_+)$ a.s. Rigorizes an observation of Kesten (1969).