Lévy Processes and Stochastic Partial Differential Equations

Davar Khoshnevisan
with M. Foondun and E. Nualart

Department of Mathematics
University of Utah
http://www.math.utah.edu/˜davar

Lévy Processes: Theory and Applications
August 13–17, 2007
Copenhagen, Denmark

Problem 1

- The stochastic heat equation:

$$\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x) \quad \forall t \in 0,\, x \in \mathbf{R}^d.$$

Problem 1

- The stochastic heat equation:

$$\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x) \quad \forall t \in 0,\, x \in \mathbf{R}^d.$$

- Question: Why $\exists$ function solutions only when $d = 1$? (Walsh, Dalang–Frangos, Dalang, Pesat–Zabczyk)

Problem 1

- The stochastic heat equation:

$$\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x) \quad \forall t \in 0,\, x \in \mathbf{R}^d.$$

- Question: Why $\exists$ function solutions only when $d = 1$?
(Walsh, Dalang–Frangos, Dalang, Pesat–Zabczyk)

Problem 1

- The stochastic heat equation:

$$\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x) \quad \forall t \in 0,\, x \in \mathbf{R}^d.$$

- Question: Why $\exists$ function solutions only when $d = 1$? (Walsh, Dalang–Frangos, Dalang, Pesat–Zabczyk)
- Answer: BM has local times only in $d = 1$.

Problem 1

- The stochastic heat equation:

$$\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x) \quad \forall t \in 0,\, x \in \mathbf{R}^d.$$

- Question: Why $\exists$ function solutions only when $d = 1$? (Walsh, Dalang–Frangos, Dalang, Pesat–Zabczyk)
- Answer: BM has local times only in $d = 1$.

Problem 1

- The stochastic heat equation:

$$\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x) \quad \forall t \in 0,\, x \in \mathbf{R}^d .$$

- Question: Why $\exists$ function solutions only when $d = 1$?
  (Walsh, Dalang–Frangos, Dalang, Pesat–Zabczyk)
- Answer: BM has local times only in $d = 1$.
  In fact, $\exists$ function solutions in dimension $2 - \alpha$ for all $\alpha \in (0\,, 2]$.

Problem 2

- Weakly interacting system of stochastic wave equations:

$$\begin{cases} \partial_{tt} u_i(t\,,x) = (\partial_{xx} u_i)(t\,,x) + \sum_{j=1}^{d} Q_{ij} \dot{W}_j(t\,,x) & \forall x \in \mathbf{R},\ t \geq 0, \\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0, \end{cases}$$

$\dot{W}_1, \ldots, \dot{W}_d :=$ i.i.d. white noises; $Q = (Q_{ij})_{i,j=1}^{d}$ invert.

Problem 2

- Weakly interacting system of stochastic wave equations:

$$\begin{cases} \partial_{tt} u_i(t\,,x) = (\partial_{xx} u_i)(t\,,x) + \sum_{j=1}^{d} Q_{ij} \dot{W}_j(t\,,x) & \forall x \in \mathbf{R},\ t \geq 0, \\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0, \end{cases}$$

  $\dot{W}_1, \ldots, \dot{W}_d :=$ i.i.d. white noises; $Q = (Q_{ij})_{i,j=1}^{d}$ invert.
- Question: When is $u(t\,,x) = 0$ for some $t > 0$ and $x \in \mathbf{R}$?

Problem 2

- Weakly interacting system of stochastic wave equations:

$$\begin{cases} \partial_{tt} u_i(t\,,x) = (\partial_{xx} u_i)(t\,,x) + \sum_{j=1}^{d} Q_{ij} \dot{W}_j(t\,,x) & \forall x \in \mathbf{R},\ t \geq 0, \\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0, & \end{cases}$$

$\dot{W}_1, \ldots, \dot{W}_d :=$ i.i.d. white noises; $Q = (Q_{ij})_{i,j=1}^{d}$ invert.

- Question: When is $u(t\,,x) = 0$ for some $t > 0$ and $x \in \mathbf{R}$?
- Answer: Iff $d < 4$.
(Orey–Pruitt, K, Dalang–Nualart; closely-related: LeGall)

The standard equation

- $\dot{W} := \{\dot{W}(t\,,x)\}_{t\geq 0, x\in\mathbf{R}^d}$ space-time white noise:

The standard equation

- $\dot{W} := \{\dot{W}(t\,,x)\}_{t\geq 0, x\in\mathbf{R}^d}$ space-time white noise:
  - $\{\dot{W}(A)\}_{A\in\mathscr{B}(\mathbf{R}_+\times\mathbf{R}^d)} :=$ a centered gaussian process.

The standard equation

- $\dot{W} := \{\dot{W}(t\,,x)\}_{t\geq 0, x\in\mathbf{R}^d}$ space-time white noise:
  - $\{\dot{W}(A)\}_{A\in\mathscr{B}(\mathbf{R}_+\times\mathbf{R}^d)}$:= a centered gaussian process.
  - $\text{Cov}(\dot{W}(A)\,,\dot{W}(B)) = |A\cap B| \quad \forall A,B\subset \mathbf{R}_+\times\mathbf{R}^d$.

The standard equation

- $\dot{W} := \{\dot{W}(t\,,x)\}_{t\geq 0, x\in\mathbf{R}^d}$ space-time white noise:
  - $\{\dot{W}(A)\}_{A\in\mathscr{B}(\mathbf{R}_+\times\mathbf{R}^d)}$:= a centered gaussian process.
  - $\text{Cov}(\dot{W}(A)\,,\dot{W}(B)) = |A\cap B| \quad \forall A, B\subset \mathbf{R}_+\times\mathbf{R}^d$.
  - Identification via Wiener integrals:

$$\int \phi\, d\dot{W} \simeq \int_0^\infty \int_{\mathbf{R}^d} \phi(t\,,x)\dot{W}(t\,,x)\,dx\,dt.$$

The standard equation

- $\dot{W} := \{\dot{W}(t\,,x)\}_{t\geq 0, x\in\mathbf{R}^d}$ space-time white noise:
  - $\{\dot{W}(A)\}_{A\in\mathscr{B}(\mathbf{R}_+\times\mathbf{R}^d)}$:= a centered gaussian process.
  - $\mathrm{Cov}(\dot{W}(A)\,,\dot{W}(B)) = |A\cap B| \quad \forall A, B \subset \mathbf{R}_+ \times \mathbf{R}^d$.
  - Identification via Wiener integrals:

$$\int \phi \, d\dot{W} \simeq \int_0^\infty \int_{\mathbf{R}^d} \phi(t\,,x)\dot{W}(t\,,x)\,dx\,dt.$$

- The stochastic heat equation: $\exists(?) u := u(t\,,x)$ $[t \geq 0,\, x \in \mathbf{R}^d]$:

$$\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x).$$

The standard equation

- $\dot{W} := \{\dot{W}(t\,,x)\}_{t\geq 0, x\in\mathbf{R}^d}$ space-time white noise:
  - $\{\dot{W}(A)\}_{A\in\mathscr{B}(\mathbf{R}_+\times\mathbf{R}^d)} :=$ a centered gaussian process.
  - $\text{Cov}(\dot{W}(A)\,,\dot{W}(B)) = |A\cap B| \quad \forall A,B\subset\mathbf{R}_+\times\mathbf{R}^d$.
  - Identification via Wiener integrals:

$$\int \phi d\dot{W} \simeq \int_0^\infty \int_{\mathbf{R}^d} \phi(t\,,x)\dot{W}(t\,,x)\,dx\,dt.$$

- The stochastic heat equation: $\exists(?)u := u(t\,,x)$ $[t\geq 0,\ x\in\mathbf{R}^d]$:

$$\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x).$$

- Fact 1: Function-valued solution $\exists$ iff $d=1$.

The standard equation

- $\dot{W} := \{\dot{W}(t\,,x)\}_{t\geq 0, x\in\mathbf{R}^d}$ space-time white noise:
  - $\{\dot{W}(A)\}_{A\in\mathscr{B}(\mathbf{R}_+\times\mathbf{R}^d)}$:= a centered gaussian process.
  - $\text{Cov}(\dot{W}(A)\,,\dot{W}(B)) = |A\cap B| \quad \forall A,B\subset\mathbf{R}_+\times\mathbf{R}^d$.
  - Identification via Wiener integrals:

$$\int \phi d\dot{W} \simeq \int_0^\infty \int_{\mathbf{R}^d} \phi(t\,,x)\dot{W}(t\,,x)\,dx\,dt.$$

- The stochastic heat equation: $\exists(?) u := u(t\,,x)$ $[t\geq 0,\, x\in\mathbf{R}^d]$:

$$\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x).$$

- Fact 1: Function-valued solution $\exists$ iff $d=1$.
- Rough explanation: Δ_x smooths; $\dot{W}$ makes rough.

The heat equation

- $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x).$

The heat equation

- $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x)$.
- Consider $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + f(t\,,x)$
  [f nice]

The heat equation

- $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x)$.
- Consider $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + f(t\,,x)$
  [f nice]
- Let

$$Q_t(y) := \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{\|y\|^2}{4t}\right).$$

The heat equation

- $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x)$.
- Consider $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + f(t\,,x)$
  [f nice]
- Let

$$Q_t(y) := \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{\|y\|^2}{4t}\right).$$

- Solution:

$$u(t\,,x) = \int_0^t \int_{\mathbf{R}^d} Q_{t-s}(x-y) f(s\,,y)\, dy\, ds.$$

The heat equation

- $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x)$.
- Consider $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + f(t\,,x)$
  [f nice]
- Let
$$Q_t(y) := \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{\|y\|^2}{4t}\right).$$
- Solution:
$$u(t\,,x) = \int_0^t \int_{\mathbf{R}^d} Q_{t-s}(x-y) f(s\,,y)\, dy\, ds.$$
- Apply to "$f := \dot{W}$." [Mild solution; Walsh, 1986]

The stochastic heat equation

- $\partial_t u(t,x) = (\Delta_x u)(t,x) + \dot{W}(t,x).$

The stochastic heat equation

- $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x)$.
- Mild solution: If $\exists$, then

$$u(t\,,x) = \int_0^t \int_{\mathbf{R}^d} Q_{t-s}(x-y)\dot{W}(dy\,ds).$$

The stochastic heat equation

- $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x)$.
- Mild solution: If $\exists$, then

$$u(t\,,x) = \int_0^t \int_{\mathbf{R}^d} Q_{t-s}(x-y)\dot{W}(dy\,ds).$$

- Weak solution: A family $u(t\,,\varphi)$, for nicely tempered φ:

$$u(t\,,\varphi) := \int_0^t \int_{\mathbf{R}^d} (Q_{t-s} * \varphi)(y)\dot{W}(dy\,ds).$$

"$u(t\,,\varphi) = \int_{\mathbf{R}^d} u(t\,,x)\varphi(x)\,dx$."

The stochastic heat equation

- $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x)$.
- Mild solution: If $\exists$, then

$$u(t\,,x) = \int_0^t \int_{\mathbf{R}^d} Q_{t-s}(x-y)\dot{W}(dy\,ds).$$

- Weak solution: A family $u(t\,,\varphi)$, for nicely tempered φ:

$$u(t\,,\varphi) := \int_0^t \int_{\mathbf{R}^d} (Q_{t-s} * \varphi)(y)\dot{W}(dy\,ds).$$

"$u(t\,,\varphi) = \int_{\mathbf{R}^d} u(t\,,x)\varphi(x)\,dx$."

- $\varphi \mapsto u(t\,,\varphi)$ is a linear gaussian distribution [Itô; Menshov].

The stochastic heat equation

- $\partial_t u(t\,,x) = (\Delta_x u)(t\,,x) + \dot{W}(t\,,x)$.
- Mild solution: If $\exists$, then

$$u(t\,,x) = \int_0^t \int_{\mathbf{R}^d} Q_{t-s}(x-y)\dot{W}(dy\,ds).$$

- Weak solution: A family $u(t\,,\varphi)$, for nicely tempered φ:

$$u(t\,,\varphi) := \int_0^t \int_{\mathbf{R}^d} (Q_{t-s} * \varphi)(y)\dot{W}(dy\,ds).$$

"$u(t\,,\varphi) = \int_{\mathbf{R}^d} u(t\,,x)\varphi(x)\,dx$."

- $\varphi \mapsto u(t\,,\varphi)$ is a linear gaussian distribution [Itô; Menshov].
- Need:

$$\mathrm{E}\left(\left|u(t\,,\varphi)\right|^2\right) = \int_0^t \int_{\mathbf{R}^d} \left|(Q_{t-s} * \varphi)(y)\right|^2 dy\,ds < \infty.$$

The stochastic heat equation

- Plancherel's theorem:

$$\int_0^t \int_{\mathbf{R}^d} \left| (Q_{t-s} * \varphi)(y) \right|^2 dy\, ds$$

The stochastic heat equation

- Plancherel's theorem:

$$\int_0^t \int_{\mathbf{R}^d} \left|(Q_{t-s} * \varphi)(y)\right|^2 dy\, ds$$

The stochastic heat equation

- Plancherel's theorem:

$$\int_0^t \int_{\mathbf{R}^d} \left| (Q_{t-s} * \varphi)(y) \right|^2 dy\, ds$$
$$= \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbf{R}^d} e^{-2\|\xi\|^2(t-s)} \left| \hat{\varphi}(\xi) \right|^2 d\xi\, ds$$

The stochastic heat equation

- Plancherel's theorem:

$$\int_0^t \int_{\mathbf{R}^d} \left|(Q_{t-s} * \varphi)(y)\right|^2 dy\, ds$$
$$= \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbf{R}^d} e^{-2\|\xi\|^2(t-s)} \left|\hat{\varphi}(\xi)\right|^2 d\xi\, ds$$
$$= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \left|\hat{\varphi}(\xi)\right|^2 \frac{1 - e^{-2t\|\xi\|^2}}{2\|\xi\|^2}\, d\xi.$$

The stochastic heat equation

- Plancherel's theorem:

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^d} \left|(Q_{t-s} * \varphi)(y)\right|^2 dy\, ds \\
&= \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbf{R}^d} e^{-2\|\xi\|^2(t-s)} \left|\hat{\varphi}(\xi)\right|^2 d\xi\, ds \\
&= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \left|\hat{\varphi}(\xi)\right|^2 \frac{1 - e^{-2t\|\xi\|^2}}{2\|\xi\|^2}\, d\xi.
\end{aligned}$$

- $\therefore$ weak solution $\exists$ iff $\varphi \in H_{-2}(\mathbf{R}^d)$. In fact,

$$\mathrm{E}\left(\left|u(t\,,\varphi)\right|^2\right) \asymp t \int_{\mathbf{R}^d} \frac{\left|\hat{\varphi}(\xi)\right|^2}{1 + t^2\|\xi\|^2}\, d\xi.$$

The stochastic heat equation

- Plancherel's theorem:

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^d} \left|(Q_{t-s} * \varphi)(y)\right|^2 dy\, ds \\
&= \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbf{R}^d} e^{-2\|\xi\|^2(t-s)} \left|\hat{\varphi}(\xi)\right|^2 d\xi\, ds \\
&= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \left|\hat{\varphi}(\xi)\right|^2 \frac{1 - e^{-2t\|\xi\|^2}}{2\|\xi\|^2}\, d\xi.
\end{aligned}$$

- $\therefore$ weak solution $\exists$ iff $\varphi \in H_{-2}(\mathbf{R}^d)$. In fact,

$$\mathrm{E}\left(\left|u(t\,,\varphi)\right|^2\right) \asymp t \int_{\mathbf{R}^d} \frac{\left|\hat{\varphi}(\xi)\right|^2}{1 + t^2\|\xi\|^2}\, d\xi.$$

- $\therefore$ mild solution $\exists$ iff $\delta_x \in H_{-2}(\mathbf{R}^d)$ $\forall x \in \mathbf{R}^d$

The stochastic heat equation

- Plancherel's theorem:

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^d} \left|(Q_{t-s} * \varphi)(y)\right|^2 dy\, ds \\
&= \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbf{R}^d} e^{-2\|\xi\|^2(t-s)} \left|\hat{\varphi}(\xi)\right|^2 d\xi\, ds \\
&= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \left|\hat{\varphi}(\xi)\right|^2 \frac{1 - e^{-2t\|\xi\|^2}}{2\|\xi\|^2}\, d\xi.
\end{aligned}$$

- $\therefore$ weak solution $\exists$ iff $\varphi \in H_{-2}(\mathbf{R}^d)$. In fact,

$$\mathrm{E}\left(\left|u(t\,,\varphi)\right|^2\right) \asymp t \int_{\mathbf{R}^d} \frac{|\hat{\varphi}(\xi)|^2}{1 + t^2\|\xi\|^2}\, d\xi.$$

- $\therefore$ mild solution $\exists$ iff $\delta_x \in H_{-2}(\mathbf{R}^d)$ $\forall x \in \mathbf{R}^d$

The stochastic heat equation

- Plancherel's theorem:

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^d} \left|(Q_{t-s} * \varphi)(y)\right|^2 dy\, ds \\
&= \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbf{R}^d} e^{-2\|\xi\|^2(t-s)} \left|\hat{\varphi}(\xi)\right|^2 d\xi\, ds \\
&= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \left|\hat{\varphi}(\xi)\right|^2 \frac{1 - e^{-2t\|\xi\|^2}}{2\|\xi\|^2}\, d\xi.
\end{aligned}$$

- $\therefore$ weak solution $\exists$ iff $\varphi \in H_{-2}(\mathbf{R}^d)$. In fact,

$$\mathrm{E}\left(\left|u(t\,,\varphi)\right|^2\right) \asymp t \int_{\mathbf{R}^d} \frac{\left|\hat{\varphi}(\xi)\right|^2}{1 + t^2\|\xi\|^2}\, d\xi.$$

- $\therefore$ mild solution $\exists$ iff $\delta_x \in H_{-2}(\mathbf{R}^d)$ $\forall x \in \mathbf{R}^d \Leftrightarrow d = 1$. ■

An explanation

- (Dalang–Frangos) Replace $\dot W(t,x)$ by $\dot F(t,x)$, where $\dot F$ is a centered gaussian noise with $\text{Cov}(\int \phi \, d\dot F \, , \, \int \psi \, d\dot F) =$

$$\iint \phi(t,x)\underbrace{(s \wedge t)\kappa(\|x-y\|)}_{\Sigma(s,t,\|x-y\|)}\psi(s,y)\,dt\,dx\,ds\,dy.$$

An explanation

- (Dalang–Frangos) Replace $\dot{W}(t\,,x)$ by $\dot{F}(t\,,x)$, where $\dot{F}$ is a centered gaussian noise with $\text{Cov}(\int \phi\, d\dot{F}\ ,\ \int \psi\, d\dot{F}) =$

$$\iint \phi(t\,,x)\underbrace{(s \wedge t)\kappa(\|x-y\|)}_{\Sigma(s\,,t\,,\|x-y\|)}\psi(s\,,y)\, dt\, dx\, ds\, dy\,.$$

- (Dalang–Frangos, Pesat–Zabczyk, Dalang) There can $\exists$ solutions for $d > 1$, depending on κ [NASC].

An explanation

- (Dalang–Frangos) Replace $\dot{W}(t\,,x)$ by $\dot{F}(t\,,x)$, where $\dot{F}$ is a centered gaussian noise with $\text{Cov}(\int \phi\, d\dot{F}\ ,\ \int \psi\, d\dot{F}) =$

$$\iint \phi(t\,,x)\underbrace{(s \wedge t)\kappa(\|x-y\|)}_{\Sigma(s\,,t\,,\|x-y\|)}\psi(s\,,y)\,dt\,dx\,ds\,dy\,.$$

- (Dalang–Frangos, Pesat–Zabczyk, Dalang) There can $\exists$ solutions for $d > 1$, depending on κ [NASC].
- Explains the roughening effect of white noise. [analytic]

An explanation

- (Dalang–Frangos) Replace $\dot{W}(t\,,x)$ by $\dot{F}(t\,,x)$, where $\dot{F}$ is a centered gaussian noise with $\text{Cov}(\int \phi\, d\dot{F}\ ,\ \int \psi\, d\dot{F}) =$

$$\iint \phi(t\,,x)\underbrace{(s \wedge t)\kappa(\|x-y\|)}_{\Sigma(s\,,t\,,\|x-y\|)}\psi(s\,,y)\, dt\, dx\, ds\, dy\,.$$

- (Dalang–Frangos, Pesat–Zabczyk, Dalang) There can $\exists$ solutions for $d > 1$, depending on κ [NASC].
- Explains the roughening effect of white noise. [analytic]
- We propose to explain the smoothing effect of Δ_x. [probabilistic]

Lévy processes

- $L := L^2$-generator of a Lévy process X in $\mathbf{R}^d$.

Lévy processes

- $L := L^2$-generator of a Lévy process X in $\mathbf{R}^d$.
- Normalization: $\mathrm{E}\exp(i\xi\cdot X(t)) = \exp(-t\Psi(\xi))$, $\hat{L}(\xi) = -\Psi(\xi)$. That is,

$$\int_{\mathbf{R}^d} f(x)(Lg)(x)\,dx = -\int_{\mathbf{R}^d} \overline{\hat{f}(\xi)}\,\hat{g}(\xi)\Psi(\xi)\,d\xi.$$

Lévy processes

- $L := L^2$-generator of a Lévy process X in $\mathbf{R}^d$.
- Normalization: $\mathrm{E}\exp(i\xi \cdot X(t)) = \exp(-t\Psi(\xi))$, $\hat{L}(\xi) = -\Psi(\xi)$. That is,

$$\int_{\mathbf{R}^d} f(x)(Lg)(x)\,dx = -\int_{\mathbf{R}^d} \overline{\hat{f}(\xi)}\,\hat{g}(\xi)\Psi(\xi)\,d\xi.$$

- $\mathrm{Dom}(L) := \{f \in L^2(\mathbf{R}^d) : \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 \mathrm{Re}\Psi(\xi)\,d\xi < \infty\}$.

Lévy processes

- $L :=$ L^2-generator of a Lévy process X in $\mathbf{R}^d$.
- Normalization: $\mathrm{E}\exp(i\xi \cdot X(t)) = \exp(-t\Psi(\xi))$, $\hat{L}(\xi) = -\Psi(\xi)$. That is,

$$\int_{\mathbf{R}^d} f(x)(Lg)(x)\,dx = -\int_{\mathbf{R}^d} \overline{\hat{f}(\xi)}\,\hat{g}(\xi)\Psi(\xi)\,d\xi.$$

- $\mathrm{Dom}(L) := \{f \in L^2(\mathbf{R}^d) : \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 \mathrm{Re}\Psi(\xi)\,d\xi < \infty\}$.
- $X' :=$ an independent copy of X;

Lévy processes

- $L :=$ L^2-generator of a Lévy process X in $\mathbf{R}^d$.
- Normalization: $\mathrm{E}\exp(i\xi \cdot X(t)) = \exp(-t\Psi(\xi))$, $\hat{L}(\xi) = -\Psi(\xi)$. That is,

$$\int_{\mathbf{R}^d} f(x)(Lg)(x)\,dx = -\int_{\mathbf{R}^d} \overline{\hat{f}(\xi)}\,\hat{g}(\xi)\Psi(\xi)\,d\xi.$$

- $\mathrm{Dom}(L) := \{f \in L^2(\mathbf{R}^d) : \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 \mathrm{Re}\Psi(\xi)\,d\xi < \infty\}$.
- $X' :=$ an independent copy of X;

Lévy processes

- $L := L^2$-generator of a Lévy process X in $\mathbf{R}^d$.
- Normalization: $\mathrm{E}\exp(i\xi \cdot X(t)) = \exp(-t\Psi(\xi))$, $\hat{L}(\xi) = -\Psi(\xi)$. That is,

$$\int_{\mathbf{R}^d} f(x)(Lg)(x)\,dx = -\int_{\mathbf{R}^d} \overline{\hat{f}(\xi)}\,\hat{g}(\xi)\Psi(\xi)\,d\xi.$$

- $\mathrm{Dom}(L) := \{f \in L^2(\mathbf{R}^d) : \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 \mathrm{Re}\Psi(\xi)\,d\xi < \infty\}$.
- $X' :=$ an independent copy of X; $\bar{X}(t) := X(t) - X'(t)$. [Lévy]

Lévy processes

- L := L^2-generator of a Lévy process X in $\mathbf{R}^d$.
- Normalization: $\mathrm{E}\exp(i\xi \cdot X(t)) = \exp(-t\Psi(\xi))$, $\hat{L}(\xi) = -\Psi(\xi)$. That is,

$$\int_{\mathbf{R}^d} f(x)(Lg)(x)\,dx = -\int_{\mathbf{R}^d} \overline{\hat{f}(\xi)}\,\hat{g}(\xi)\Psi(\xi)\,d\xi.$$

- $\mathrm{Dom}(L) := \{f \in L^2(\mathbf{R}^d) : \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 \mathrm{Re}\Psi(\xi)\,d\xi < \infty\}$.
- X' := an independent copy of X; $\bar{X}(t) := X(t) - X'(t)$. [Lévy]
- $\bar{X}$ is a Lévy process with char. exponent $2\mathrm{Re}\Psi$.

Local times

- $\lambda_t^x :=$ local time of $\bar{X}$, at place x and time t, when it exists.

Local times

- $\lambda_t^x :=$ local time of $\bar{X}$, at place x and time t, when it exists.
- Defining property: $\forall$ Borel meas. $f : \mathbf{R}^d \to \mathbf{R}_+$, $t > 0$,

$$\int_0^t f(\bar{X}(s))\,ds = \int_{\mathbf{R}^d} f(x)\lambda_t^x\,dx \qquad \text{a.s.}$$

Local times

- $\lambda_t^x :=$ local time of $\bar{X}$, at place x and time t, when it exists.
- Defining property: $\forall$ Borel meas. $f : \mathbf{R}^d \to \mathbf{R}_+$, $t > 0$,

$$\int_0^t f(\bar{X}(s))\, ds = \int_{\mathbf{R}^d} f(x) \lambda_t^x \, dx \qquad \text{a.s.}$$

- $\lambda_t^x = O_t(dx)/dx$, where $O_t(E) := \int_0^t \mathbf{1}_E(\bar{X}(s))\, ds$.

Local times

- $\lambda_t^x :=$ local time of $\bar{X}$, at place x and time t, when it exists.
- Defining property: $\forall$ Borel meas. $f : \mathbf{R}^d \to \mathbf{R}_+$, $t > 0$,

$$\int_0^t f(\bar{X}(s))\, ds = \int_{\mathbf{R}^d} f(x)\lambda_t^x\, dx \qquad \text{a.s.}$$

- $\lambda_t^x = O_t(dx)/dx$, where $O_t(E) := \int_0^t \mathbf{1}_E(\bar{X}(s))\, ds$.

Theorem (Hawkes)

$\{\lambda_t^x\}_{t \geq 0, x \in \mathbf{R}^d}$ *exists iff*

$$\int_{\mathbf{R}^d} \frac{d\xi}{1 + \mathrm{Re}\Psi(\xi)} < \infty.$$

Examples and remarks

- Hawkes' condition: $\int_{\mathbf{R}^d} (1 + \text{Re}\Psi(\xi))^{-1} \, d\xi < \infty$.

Examples and remarks

- Hawkes' condition: $\int_{\mathbf{R}^d} (1 + \text{Re}\Psi(\xi))^{-1} d\xi < \infty$.
- $\Psi(\xi) = O(\|\xi\|^2)$ [Bochner]. Therefore, local times can $\exists$ only when $d = 1$, if at all.

Examples and remarks

- Hawkes' condition: $\int_{\mathbf{R}^d} (1 + \text{Re}\Psi(\xi))^{-1}\, d\xi < \infty$.
- $\Psi(\xi) = O(\|\xi\|^2)$ [Bochner]. Therefore, local times can $\exists$ only when $d = 1$, if at all.
- If $d = 1$ and $\Psi(\xi) = |\xi|^\alpha$ then local times exist iff $\alpha > 1$.

Examples and remarks

- Hawkes' condition: $\int_{\mathbf{R}^d} (1 + \mathrm{Re}\Psi(\xi))^{-1}\, d\xi < \infty$.
- $\Psi(\xi) = O(\|\xi\|^2)$ [Bochner]. Therefore, local times can $\exists$ only when $d = 1$, if at all.
- If $d = 1$ and $\Psi(\xi) = |\xi|^\alpha$ then local times exist iff $\alpha > 1$.
- For general (nonsymmetric) Lévy processes, the Hawkes condition is that

$$\int_{-\infty}^{\infty} \mathrm{Re}\left(\frac{1}{1 + \Psi(\xi)}\right) d\xi < \infty.$$

Examples and remarks

- Hawkes' condition: $\int_{\mathbf{R}^d} (1 + \text{Re}\Psi(\xi))^{-1} \, d\xi < \infty$.
- $\Psi(\xi) = O(\|\xi\|^2)$ [Bochner]. Therefore, local times can $\exists$ only when $d = 1$, if at all.
- If $d = 1$ and $\Psi(\xi) = |\xi|^\alpha$ then local times exist iff $\alpha > 1$.
- For general (nonsymmetric) Lévy processes, the Hawkes condition is that

$$\int_{-\infty}^{\infty} \text{Re}\left(\frac{1}{1 + \Psi(\xi)}\right) d\xi < \infty.$$

- Suppose $d = 1$ and $\Psi(\xi) = |\xi|(1 + ic\,\text{sgn}|\xi| \log|\xi|)$ for $0 \le |c| \le 2/\pi$. Then local times exist iff $c \neq 0$.

A heat equation

- Consider the heat equation

$$\partial_t u(t\,,x) = (L_x u)(t\,,x) + \dot{W}(t\,,x),$$

where L_x is the generator of a Lévy process on $\mathbf{R}^d$, acting on the variable x.

A heat equation

- Consider the heat equation

$$\partial_t u(t\,,x) = (L_x u)(t\,,x) + \dot{W}(t\,,x),$$

where L_x is the generator of a Lévy process on $\mathbf{R}^d$, acting on the variable x.

- We call this the heat equation for L.

A heat equation

- Consider the heat equation

$$\partial_t u(t\,,x) = (L_x u)(t\,,x) + \dot{W}(t\,,x),$$

where L_x is the generator of a Lévy process on $\mathbf{R}^d$, acting on the variable x.
- We call this the heat equation for L.
- Fundamental questions:

A heat equation

- Consider the heat equation

$$\partial_t u(t\,,x) = (L_x u)(t\,,x) + \dot{W}(t\,,x),$$

  where L_x is the generator of a Lévy process on $\mathbf{R}^d$, acting on the variable x.
- We call this the heat equation for L.
- Fundamental questions:
  - When is there a function-valued solution? [General answer by Dalang]

A heat equation

- Consider the heat equation

$$\partial_t u(t\,,x) = (L_x u)(t\,,x) + \dot{W}(t\,,x),$$

where L_x is the generator of a Lévy process on $\mathbf{R}^d$, acting on the variable x.
- We call this the heat equation for L.
- Fundamental questions:
  - When is there a function-valued solution? [General answer by Dalang]
  - When is there a continuous solution?

A heat equation

- Consider the heat equation

$$\partial_t u(t,x) = (L_x u)(t,x) + \dot{W}(t,x),$$

where L_x is the generator of a Lévy process on $\mathbf{R}^d$, acting on the variable x.

- We call this the heat equation for L.
- Fundamental questions:
  - When is there a function-valued solution? [General answer by Dalang]
  - When is there a continuous solution?
  - When is there a Hölder-continuous solution?

A heat equation

- Consider the heat equation

$$\partial_t u(t\,,x) = (L_x u)(t\,,x) + \dot{W}(t\,,x),$$

  where L_x is the generator of a Lévy process on $\mathbf{R}^d$, acting on the variable x.
- We call this the heat equation for L.
- Fundamental questions:
  - When is there a function-valued solution? [General answer by Dalang]
  - When is there a continuous solution?
  - When is there a Hölder-continuous solution?
  - Etc.

A heat equation

Theorem (K–Foondun–Nualart)

Let u denote the weak solution to the heat equation for L. Then, for all tempered functions φ and all $t, \lambda > 0$,

$$\frac{1-e^{-2\lambda t}}{2}\mathscr{E}_\lambda(\varphi\,,\varphi) \le \mathrm{E}\left(|u(t\,,\varphi)|^2\right) \le \frac{e^{2\lambda t}}{2}\mathscr{E}_\lambda(\varphi\,,\varphi),$$

where

$$\mathscr{E}_\lambda(\varphi\,,\psi) := \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \frac{\hat\varphi(\xi)\overline{\hat\psi(\xi)}}{\lambda + \mathrm{Re}\Psi(\xi)}\,d\xi.$$

A heat equation

Theorem (K–Foondun–Nualart)

Let u denote the weak solution to the heat equation for L. Then, for all tempered functions φ and all $t, \lambda > 0$,

$$\frac{1 - e^{-2\lambda t}}{2}\mathscr{E}_\lambda(\varphi, \varphi) \leq \mathrm{E}\left(|u(t, \varphi)|^2\right) \leq \frac{e^{2\lambda t}}{2}\mathscr{E}_\lambda(\varphi, \varphi),$$

where

$$\mathscr{E}_\lambda(\varphi, \psi) := \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \frac{\hat{\varphi}(\xi)\overline{\hat{\psi}(\xi)}}{\lambda + \mathrm{Re}\Psi(\xi)}\, d\xi.$$

Corollary

$\exists$ function-valued solutions iff $\bar{X}$ has local times. The solution is continuous iff $x \mapsto \lambda_t^x$ is.

A heat equation

- In fact, many [virtually all] of the properties of $x \mapsto u(t\,,x)$ are inherited from $x \mapsto \lambda_t^x$ and vice versa:

A heat equation

- In fact, many [virtually all] of the properties of $x \mapsto u(t\,,x)$ are inherited from $x \mapsto \lambda_t^x$ and vice versa:
  - Existence and continuity.

A heat equation

- In fact, many [virtually all] of the properties of $x \mapsto u(t\,,x)$ are inherited from $x \mapsto \lambda_t^x$ and vice versa:
  - Existence and continuity.
  - Hölder continuity.

A heat equation

- In fact, many [virtually all] of the properties of $x \mapsto u(t\,,x)$ are inherited from $x \mapsto \lambda_t^x$ and vice versa:
  - Existence and continuity.
  - Hölder continuity.
  - p-variation of the paths . . . .

A heat equation

- In fact, many [virtually all] of the properties of $x \mapsto u(t\,,x)$ are inherited from $x \mapsto \lambda_t^x$ and vice versa:
  - Existence and continuity.
  - Hölder continuity.
  - p-variation of the paths . . . .
- $\exists$ an embedding of the isomorphism theorem? [Dynkin; Brydges–Fröhlich–Spencer]

A heat equation

- In fact, many [virtually all] of the properties of $x \mapsto u(t\,,x)$ are inherited from $x \mapsto \lambda_t^x$ and vice versa:
  - Existence and continuity.
  - Hölder continuity.
  - p-variation of the paths … .
- $\exists$ an embedding of the isomorphism theorem? [Dynkin; Brydges–Fröhlich–Spencer]
- A final **Theorem** (K–Foondun–Nualart): $t \mapsto u(t\,,\varphi)$ has a continuous version iff

$$\int_1^\infty \frac{\mathscr{E}_\lambda(\varphi\,,\varphi)}{\lambda\sqrt{|\log\lambda|}}\,d\lambda < \infty.$$

Solutions in dimension $2 - \epsilon$

- Recall:

$$\partial_t u(t\,,x) = (L_x u)(t\,,x) + \dot{W}(t\,,x). \qquad \text{(HE)}$$

Solutions in dimension $2 - \epsilon$

- Recall:

$$\partial_t u(t\,,x) = (L_x u)(t\,,x) + \dot{W}(t\,,x). \qquad \text{(HE)}$$

- We chose L to be the generator of a Lévy process only because there we have NASC.

Solutions in dimension $2 - \epsilon$

- Recall:

$$\partial_t u(t\,,x) = (L_x u)(t\,,x) + \dot{W}(t\,,x). \tag{HE}$$

- We chose L to be the generator of a Lévy process only because there we have NASC.
- Could have $L :=$ generator of a nice Markov process.

Solutions in dimension $2 - \epsilon$

- Recall:

$$\partial_t u(t\,,x) = (L_x u)(t\,,x) + \dot{W}(t\,,x). \qquad \text{(HE)}$$

- We chose L to be the generator of a Lévy process only because there we have NASC.
- Could have $L :=$ generator of a nice Markov process.

Theorem (K–Foondun–Nualart)

Let $L :=$ Laplacian on a "nice" fractal of $\dim_{\mathrm{H}} = 2 - \alpha$ for $\alpha \in (0\,,2]$. Then (HE) has function solutions that are in fact Hölder continuous.

System of wave equations

- Let $\dot{L} :=$ be space–time Lévy noise with values in $\mathbf{R}^d$.

System of wave equations

- Let $\dot{L} :=$ be space–time Lévy noise with values in $\mathbf{R}^d$.
- $\mathrm{E}\exp(i\xi \cdot \dot{L}(A)) = \exp(-|A|\Psi(\xi))$ for $\xi \in \mathbf{R}^d$ and $A \in \mathscr{B}(\mathbf{R}_+ \times \mathbf{R})$.

System of wave equations

- Let $\dot{L}$:= be space–time Lévy noise with values in $\mathbf{R}^d$.
- $\mathrm{E}\exp(i\xi \cdot \dot{L}(A)) = \exp(-|A|\Psi(\xi))$ for $\xi \in \mathbf{R}^d$ and $A \in \mathscr{B}(\mathbf{R}_+ \times \mathbf{R})$.
- Assume:

System of wave equations

- Let $\dot{L} :=$ be space–time Lévy noise with values in $\mathbf{R}^d$.
- $\mathrm{E}\exp(i\xi \cdot \dot{L}(A)) = \exp(-|A|\Psi(\xi))$ for $\xi \in \mathbf{R}^d$ and $A \in \mathscr{B}(\mathbf{R}_+ \times \mathbf{R})$.
- Assume:
  - Ψ is nonnegative real.

System of wave equations

- Let $\dot{L}$:= be space–time Lévy noise with values in $\mathbf{R}^d$.
- $\mathrm{E}\exp(i\xi\cdot\dot{L}(A)) = \exp(-|A|\Psi(\xi))$ for $\xi\in\mathbf{R}^d$ and $A\in\mathscr{B}(\mathbf{R}_+\times\mathbf{R})$.
- Assume:
  - Ψ is nonnegative real.
  - $\forall a>0\,\exists A_a>0$ such that

$$\Psi(a\xi)\geq A_a\Psi(\xi).$$

[stable like]

System of wave equations

- Let $\dot{L} :=$ be space–time Lévy noise with values in $\mathbf{R}^d$.
- $\mathrm{E}\exp(i\xi \cdot \dot{L}(A)) = \exp(-|A|\Psi(\xi))$ for $\xi \in \mathbf{R}^d$ and $A \in \mathscr{B}(\mathbf{R}_+ \times \mathbf{R})$.
- Assume:
  - Ψ is nonnegative real.
  - $\forall a > 0 \, \exists A_a > 0$ such that

$$\Psi(a\xi) \geq A_a \Psi(\xi).$$

  [stable like]
- Gauge function $\exists$ and is finite, where

$$\Phi(\lambda) := \int_{\mathbf{R}^d} e^{-\lambda\Psi(\xi)}\, d\xi \qquad {}^{\forall}\lambda > 0.$$

Zeros of the solution

$$\begin{cases} \partial_{tt} u_i(t\,,x) = \partial_{xx} u_i(x\,,t) + \dot{L}_i(t\,,x), \\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0. \end{cases}$$

Zeros of the solution

$$\begin{cases} \partial_{tt} u_i(t\,,x) = \partial_{xx} u_i(x\,,t) + \dot{L}_i(t\,,x), \\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0. \end{cases}$$

Theorem (K–Nualart)

TFAE:

Zeros of the solution

$$\left[\begin{array}{l}\partial_{tt} u_i(t\,,x) = \partial_{xx} u_i(x\,,t) + \dot{L}_i(t\,,x),\\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0.\end{array}\right.$$

Theorem (K–Nualart)

TFAE:

1. *With positive probab.* $u(t\,,x) = 0$ *for some* $t > 0$ *and* $x \in \mathbf{R}$.

Zeros of the solution

$$\begin{cases} \partial_{tt} u_i(t\,,x) = \partial_{xx} u_i(x\,,t) + \dot{L}_i(t\,,x), \\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0. \end{cases}$$

Theorem (K–Nualart)

TFAE:

1. *With positive probab.* $u(t\,,x) = 0$ *for some* $t > 0$ *and* $x \in \mathbf{R}$.
2. *Almost surely,* $u(t\,,x) = 0$ *for some* $t > 0$ *and* $x \in \mathbf{R}$.

Zeros of the solution

$$\begin{cases} \partial_{tt} u_i(t\,,x) = \partial_{xx} u_i(x\,,t) + \dot{L}_i(t\,,x), \\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0. \end{cases}$$

Theorem (K–Nualart)

TFAE:

1. *With positive probab.* $u(t\,,x) = 0$ *for some* $t > 0$ *and* $x \in \mathbf{R}$.
2. *Almost surely,* $u(t\,,x) = 0$ *for some* $t > 0$ *and* $x \in \mathbf{R}$.
3. $\int_0^1 \lambda \Phi(\lambda)\, d\lambda < \infty$.

Zeros of the solution

$$\begin{cases} \partial_{tt} u_i(t\,,x) = \partial_{xx} u_i(x\,,t) + \dot{L}_i(t\,,x), \\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0. \end{cases}$$

Theorem (K–Nualart)

TFAE:

1. *With positive probab.* $u(t\,,x) = 0$ *for some* $t > 0$ *and* $x \in \mathbf{R}$.
2. *Almost surely,* $u(t\,,x) = 0$ *for some* $t > 0$ *and* $x \in \mathbf{R}$.
3. $\int_0^1 \lambda \Phi(\lambda)\, d\lambda < \infty$.
4. *If* $\int_0^1 \lambda \Phi(\lambda)\, d\lambda < \infty$, *then a.s.,*

$$\dim_{_{\rm H}} u^{-1}\{0\} = 2 - \limsup_{\lambda \downarrow 0} \frac{\log \Phi(\lambda)}{\log(1/\lambda)}.$$

Zeros of the solution

$$\begin{cases} \partial_{tt} u_i(t\,,x) = \partial_{xx} u_i(x\,,t) + \dot{L}_i(t\,,x), \\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0. \end{cases}$$

Zeros of the solution

$$\begin{cases} \partial_{tt} u_i(t\,,x) = \partial_{xx} u_i(x\,,t) + \dot{L}_i(t\,,x), \\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0. \end{cases}$$

Example

Suppose $\dot{L}_1, \ldots, \dot{L}_d$ are independent, $\dot{L}_j = \text{stable}(\alpha_j)$. Then:

Zeros of the solution

$$\begin{cases} \partial_{tt} u_i(t\,,x) = \partial_{xx} u_i(x\,,t) + \dot{L}_i(t\,,x), \\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0. \end{cases}$$

Example

Suppose $\dot{L}_1, \ldots, \dot{L}_d$ are independent, $\dot{L}_j = \text{stable}(\alpha_j)$. Then:

1. u has zeros iff $\sum_{j=1}^{d}(1/\alpha_j) < 2$.

Zeros of the solution

$$\begin{cases} \partial_{tt} u_i(t\,,x) = \partial_{xx} u_i(x\,,t) + \dot{L}_i(t\,,x), \\ u_i(0\,,x) = \partial_t u_i(0\,,x) = 0. \end{cases}$$

Example

Suppose $\dot{L}_1, \ldots, \dot{L}_d$ are independent, $\dot{L}_j = \text{stable}(\alpha_j)$. Then:

1. u has zeros iff $\sum_{j=1}^{d}(1/\alpha_j) < 2$.
2. If $\sum_{j=1}^{d}(1/\alpha_j) < 2$, then $\dim_{\mathrm{H}} u^{-1}\{0\} = 2 - \sum_{j=1}^{d}(1/\alpha_j)$.

Idea of proof [existence part]

- WLOG consider $u(t\,,x)$ for $t,x \geq 0$.

Idea of proof [existence part]

- WLOG consider $u(t\,,x)$ for $t,x \geq 0$.
- $\mathrm{P}\{0 \in u(G)\} \asymp \mathrm{P}\{0 \in X(G)\}$, where

$$X(t\,,x) := X_1(t) + X_2(x),$$

where X_1, X_2 are i.i.d. Lévy processes, exponent Ψ.
[additive Lévy process]

Idea of proof [existence part]

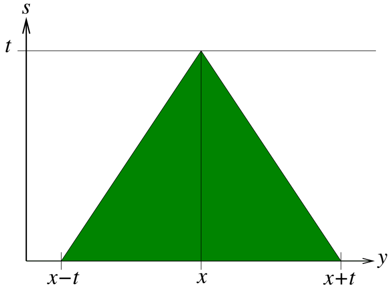
- WLOG consider $u(t\,,x)$ for $t,x \geq 0$.
- $\mathrm{P}\{0 \in u(G)\} \asymp \mathrm{P}\{0 \in X(G)\}$, where

$$X(t\,,x) := X_1(t) + X_2(x),$$

  where X_1, X_2 are i.i.d. Lévy processes, exponent Ψ.
  [additive Lévy process]
- Appeal to K–Shieh–Xiao.

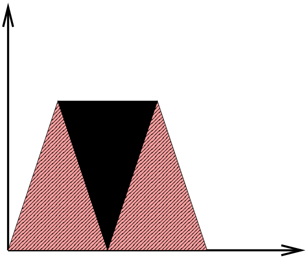
Idea of proof [second part]

$u(t\,,x) = \frac{1}{2}\dot{L}(\mathscr{C}(t\,,x))$, where $\mathscr{C}(t\,,x)$ is the “light cone” emanating from $(t\,,x)$.



Idea of proof [zero-one law part]

The zero set in the black triangle depends on the noise through its "backward light cone," shaded black/pink.



Therefore, $\mathrm{P}\{u^{-1}(\{0\}) \neq \varnothing\}$ is zero or one.