Lecture 5

Terminus: Capacity

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Recap

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- $\mathrm{Cap}_\kappa(F) := 1/\inf_{\mu \in \mathscr{P}(F)} \mathscr{E}_\kappa(\mu)$.
- Last time: If $m_d(F) = 0$ then

$$\int_{\mathbf{R}^d} \mathrm{P}_x \{X(t) \in F \text{ for some } t > 0\}\, dx \geq \mathrm{Cap}_\kappa(F).$$

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- Primary task today: Converse.

A martingale theory

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- $\mathrm{E}_{m_d}[Z \,|\, \mathscr{F}_t]$ defined as usual: For all $\mathscr{F}_t$-measurable $Z, Y \in L^2(\mathrm{P}_{m_d})$,

$$\mathrm{E}_{m_d}\left[ZY\right] = \mathrm{E}_{m_d}\left[\mathrm{E}_{m_d}\left(Z \,\middle|\, \mathscr{F}_t\right) Y\right].$$

$\exists$(!) by Radon–Nikodým and/or the projection theorem

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  $\exists$(!) by Radon–Nikodým and/or the projection theorem
- OST: If T is a bounded stopping time [$\mathscr{F}$] and $Z \in L^1(\mathrm{P}_{m_d}) \cap L^2(\mathrm{P}_{m_d})$, then

$$\mathrm{E}_{m_d}\left[\mathrm{E}_{m_d}\left(Z \,\middle|\, \mathscr{F}_T\right)\right] = \mathrm{E}_{m_d}[Z].$$

The semigroup

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$$\mathrm{E}_{m_d}[f(X(s))g(X(t))] = \int_{\mathbf{R}^d} f(z)(P_{t-s} * g)(z)\,dz.$$

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- If $0 \le s_1 < s_2 < \cdots < s_k < t$, then

$$\mathrm{E}_{m_d}\left(\prod_{j=1}^k f_j(X(s_j))\,g(X(t))\right)$$
$$= \int_{\mathbf{R}^d} \mathrm{E}\left(\prod_{j=1}^k f_j(X(s_j)+x)\,g(X(t)+x)\right) dx.$$

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A Strong Markov property

- One way:

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- More or less standard arguments $\Rightarrow \forall$ bounded stopping times S,

$$\mathrm{E}_{m_d}\left(g(X(t+S)) \,\middle|\, \mathscr{F}_S\right) = (P_t * g)(X(S)).$$

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&= e^{-n}\int_0^\infty (P_t * f)(X(S))e^{-t}\,dt\cdot\mathbf{1}_{\{S\leq n\}} \\
&= e^{-n}(U * f)(X(S))\cdot\mathbf{1}_{\{S\leq n\}}.
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- If f is smooth, then

$$\int (U * f)\, d\mu = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \overline{\hat{\mu}(\xi)}\, \hat{f}(\xi)\, \hat{U}(\xi)\, d\xi.$$

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- Set $f := \mu * \phi_\epsilon$, where $\hat{\phi}_\epsilon \geq 0$ and ϕ_ϵ is a probab. density.

The upper bound

- We obtain

$$\mathrm{P}_{m_d}\{S \leq n\} \times \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{\mu}(\xi)|^2 \; \hat{\phi}_\epsilon(\xi)\, \hat{U}(\xi)\, d\xi \leq e^n.$$

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- If $\mathrm{P}_{m_d}\{S \le n\} > 0$, then $\exists \mu \in \mathscr{P}(F)$ such that $\mathscr{E}_\kappa(\mu) < \infty$.
- $\mathrm{P}_{m_d}\{S < \infty\} > 0 \;\Rightarrow\; \mathrm{Cap}_\kappa(F) > 0$.

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- $\mathrm{P}_{m_d}\{S < \infty\} > 0 \;\Rightarrow\; \mathrm{Cap}_\kappa(F) > 0$.
- $\mathrm{P}_{m_d}\{S < \infty\} = \int_{\mathbf{R}^d} \mathrm{P}_x\{X(t) \in F \text{ for some } t > 0\}\,dx.$

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1. $\mathrm{Cap}_\kappa(F) > 0$.
2. $\int_{\mathbf{R}^d} \mathrm{P}_x\{X(t) \in F \text{ for some } t > 0\}\, dx > 0$.

To start at the origin

- By the usual MP:

$$\mathrm{P}\{X(t) \in F \text{ for some } t > s\}$$
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- Multiply by e^{-s} and integrate $[ds]$:

$$\begin{aligned}&\int_0^\infty \mathrm{P}\{X(t) \in F \text{ for some } t > s\} e^{-s}\, ds\\&= \int_{\mathbf{R}^d} \mathrm{P}_x\left\{X(t) \in F \text{ for some } t > 0\right\} U(dx).\end{aligned}$$

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Theorem (Blumenthal–Getoor, Hawkes, Hunt, ...)

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Theorem (Blumenthal–Getoor, Hawkes, Hunt, ...)

$F :=$ nonrandom compact. Suppose $U(dx) = u(x)dx$ and $u(x) > 0$ for all x. TFAE:

1. $\text{Cap}_\kappa(F) > 0$.
2. $\text{P}\{X(t) \in F \text{ for some } t > 0\} > 0$.

A capacity identity

Recall: $\mathrm{Cap}_\kappa(F) := [\inf_{\mu \in \mathscr{P}(F)} \mathscr{E}_\kappa(\mu)]^{-1}$, where

$$\mathscr{E}_\kappa(\mu) := \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \hat{\kappa}(\xi) |\hat{\mu}(\xi)|^2 \, d\xi.$$

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Final goal (K–Xiao, 2007)

Suppose $U(dx) = u(x)\,dx$. Then $\forall$ Borel F,

$$\mathrm{Cap}_\kappa(F) = \left[\inf_{\mu\in\mathscr{P}_c(F)} \iint u(x-y)\,\mu(dx)\,\mu(dy)\right]^{-1}.$$

When F is open this is much easier to prove (Hawkes).

A capacity identity

Let $v(x) := \frac{1}{2}\{u(x) + u(-x)\}$. Note that

$$\iint u(x-y)\,\mu(dx)\,\mu(dy) = \iint v(x-y)\,\mu(dx)\,\mu(dy).$$

We need some basic facts (see Problems).

- Fact 1: $\kappa = \hat{v}$ in the sense of L^2; i.e., if ϕ is well tempered, then $(\phi\,, v) = (2\pi)^{-d}(\hat{\phi}\,, \kappa)$.

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- Fact 1: $\kappa = \hat{v}$ in the sense of L^2; i.e., if ϕ is well tempered, then $(\phi\,, v) = (2\pi)^{-d}(\hat{\phi}\,, \kappa)$.
- Fact 2: u, and hence v, is lower semicontinuous.
- Fact 3: $\iint u(x-y)\,\mu(dx)\,\mu(dy) \le (2\pi)^{-d} \int_{\mathbf{R}^d} \hat{\kappa}(\xi)|\hat{\mu}(\xi)|^2\,d\xi$. Therefore, $\mathrm{Cap}_\kappa(F) \le 1/\inf \iint u(x-y)\,\mu(dx)\,\mu(dy)$.

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- WLOG $RHS > 0$; i.e.,

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- Therefore, $\int (v * \mu)\, d\mu \geq \inf (2\pi)^{-d} \int_{\mathbf{R}^d} \kappa(\xi) |\hat{\rho}(\xi)|^2\, d\xi$, where “inf” is over all measures ρ on F with mass $\in [1 - \eta\,, 1]$.

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- By scaling,
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- Let $\eta \to 0$ to finish. ■

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3. Prove Fact 3. (Hint: First replace u by $u * \phi_\epsilon$, and then use Plancherel and Fatou in this order.)