Lecture 4

More on the Range

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Summer School on Lévy Processes: Theory and Applications
August 9–12, 2007
Sandbjerg Manor, Denmark

A zero-one law

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Let ζ be an independent mean-one exponential random variable.

Corollary

$m_d(X(\mathbf{R}_+)) > 0 \Leftrightarrow \mathrm{P}\{m_d(X([0\,,\zeta])) > 0\} > 0.$

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- Integrate [dx] via Tonelli:

$$\int_{\mathbf{R}^d} \mathrm{P}\{X([0\,,\zeta]) \cap B(x\,,r)) \neq \varnothing\}\, dx \le \frac{cr^d}{U(B(0\,,r))}.$$

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- $X([0\,,\zeta])\cap B(x\,,r))\neq\varnothing$ iff $x\in X([0\,,\zeta])^r$.
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- $m_d(X([0\,,\zeta])^r) \downarrow m_d(X([0\,,\zeta]))$ since X is cadlag.

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Note Bene

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Thus, for all closed $A \subset \mathbf{R}^d$, $U(A) \leq cm_d(A)$. This proves that $U(dx) = u(x)dx$ with $\sup u \leq c$. ■

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$\therefore\ \kappa \in L^1(\mathbf{R}^d) \Rightarrow m_d(X(\mathbf{R}_+)) > 0$.

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This $\geq (1 + o(1)) \int \kappa(\xi)\, d\xi$ as $r \downarrow 0$ [Fatou]. ■

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- $\mathrm{E}_x J(f) = \int_0^\infty \mathrm{E}[f(X(t)+x)]e^{-t}\,dt.$
- $\mathrm{P}_{m_d}(A) := \int_{\mathbf{R}^d} \mathrm{P}_x(A)\,dx$; $\mathrm{E}_{m_d}(Z) := \int_{\mathbf{R}^d} \mathrm{E}_x(Z)\,dx.$

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Given a [compact] set $F \subset \mathbf{R}^d$, when is $\mathrm{P}\{X([0,1]) \cap F \neq \varnothing\} > 0$? [Is F is nonpolar?]

- A sufficient condition (F compact): Define $\forall f : \mathbf{R}^d \to \mathbf{R}_+$ meas.,

$$J(f) := \int_0^\infty f(X(t)) e^{-t}\, dt.$$

- $\mathrm{E}_x J(f) = \int_0^\infty \mathrm{E}[f(X(t)+x)] e^{-t}\, dt$.
- $\mathrm{P}_{m_d}(A) := \int_{\mathbf{R}^d} \mathrm{P}_x(A)\, dx$; $\mathrm{E}_{m_d}(Z) := \int_{\mathbf{R}^d} \mathrm{E}_x(Z)\, dx$.
- $\mathrm{E}_{m_d} J(f) = \int_{\mathbf{R}^d} f(x)\, dx = 1$ [say].

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- Exercise: Use Fubini to prove that, always,

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3. Examples.

Problems

Suppose $\mathrm{P}\{X(s) \in A\} = \int_A p_s(x)\,dx$.
[transition densities]

1. Then prove that $U(dx)/dx := u(x) = \int_0^\infty p_s(x)e^{-s}\,ds$.

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1. Then prove that $U(dx)/dx := u(x) = \int_0^\infty p_s(x)e^{-s}\,ds$.
2. Prove that $p_s \in L^2(\mathbf{R}^d)\ \forall s > 0$ iff $\int_{\mathbf{R}^d} e^{-s\mathrm{Re}\Psi(\xi)}\,d\xi < \infty\ \forall \xi \in \mathbf{R}^d$. In this case, $(s\,, x) \mapsto p_s(x)$ is uniformly continuous on $[\epsilon\,, 1/\epsilon] \times \mathbf{R}^d$, $\forall \epsilon > 0$.
3. Suppose $p_s \in L^2(\mathbf{R}^d)\ \forall s > 0$ and X is symmetric [Ψ real]. Prove that $u(x) \le u(0)$ for all $x \in \mathbf{R}^d$. Thus, $\sup u < \infty$ iff $u(0) < \infty$. Use this to prove that if X is isotropic stable(α), then $m_d(X(\mathbf{R}_+)) > 0$ iff $\alpha > d$. Use Kesten's theorem for an alternative derivation.