Lecture 3

Harmonic Analysis

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The Fourier transform on $L^1(\mathbf{R}^d)$

- If $f \in L^1(\mathbf{R}^d)$, then its Fourier transform is

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- The inversion theorem: If $f, \hat{f} \in L^1(\mathbf{R}^d)$, then a.e.,

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-ix\cdot\xi} \hat{f}(\xi)\, d\xi.$$

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- $\hat{\mu}(\xi) := \int_{\mathbf{R}^d} \exp(ix\cdot\xi)\, \mu(dx)$ for all finite Borel measures μ on $\mathbf{R}^d$.
- Parseval's identity: $\int \hat{f}(x)\, \mu(dx) = \int f(\xi)\hat{\mu}(\xi)\, d\xi$.

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[The Cauchy transform; $\exists$ a converse]

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Corollary

$\overline{\mathrm{ind}}\, U = \limsup_{\epsilon \to 0} \frac{\log W(\epsilon)}{\log \epsilon} \qquad \underline{\mathrm{ind}}\, U = \liminf_{\epsilon \to 0} \frac{\log W(\epsilon)}{\log \epsilon}$.

Fourier-analytic dimension formulas

Theorem (Kh-Xiao, 2007)

Almost surely,

$$\begin{aligned} \dim_{\text{H}} X([0\,,1]) &= \liminf_{\epsilon\to 0} \frac{\log W(\epsilon)}{\log \epsilon}, \\ \overline{\dim}_{\text{M}} X([0\,,1]) &= \limsup_{\epsilon\to 0} \frac{\log W(\epsilon)}{\log \epsilon}, \end{aligned}$$

where

$$W(\epsilon) := \int_{\mathbf{R}^d} \frac{\kappa(\xi/\epsilon)}{\prod_{j=1}^d (1+\xi_j^2)}\, d\xi, \quad \kappa(z) := \text{Re}\left(\frac{1}{1+\Psi(z)}\right).$$

Example: Brownian motion

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If $X := \mathrm{BM}(\mathbf{R}^d)$, then $\dim_{_\mathrm{H}} X([0\,,1]) = \overline{\dim}_{_\mathrm{M}} X([0\,,1]) = d \wedge 2$ a.s.

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- Integrate $[e^{-t}\,dt] \Rightarrow$

$$\frac{1}{1+\Phi(\lambda)} = \frac{W(1/\lambda)}{\pi}.$$

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Example (Horowitz, Pruitt–Taylor, Bertoin)

$X :=$ subordinator with Laplace exponent Φ; a.s.:

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Fact (Maisonneuve)

The zero-set of a rather general Markov process can be identified with the range of a subordinator; Φ is often fairly explicit.

Problems

1. $X :=$ an isotropic stable process, index $\alpha \in (0\,,2]$: $\Psi(\xi) = \lambda^2\|\xi\|^\alpha$. WLOG $\lambda = 1$. Prove: $\dim_{_H} X([0\,,1]) = \overline{\dim}_{_M} X([0\,,1]) = (d \wedge \alpha)$ a.s.

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2. A classical fact: Given any Lévy process X with characteristic exponent Ψ, $|\Psi(\xi)| = O(\|\xi\|^2)$. Let $X(t) = B(t) + Y(t)$, where $B :=$ a nondegenerate BM($\mathbf{R}^d$), and Y is an independent pure-jump Lévy process. Use the preceding fact, without proof, to deduce that $\dim_{_H} X([0\,,1]) \geq \dim_{_H} Y([0\,,1])$ and $\overline{\dim}_{_M} X([0\,,1]) \geq \overline{\dim}_{_M} Y([0\,,1]$ a.s.

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    3.2 Prove that $\dim Y([0\,,1]) \leq \dim X([0\,,1])$ a.s., where dim stands for either $\dim_{_H}$ or $\overline{\dim}_{_M}$.

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   [Both inequalities are strict.]
2. Suppose X is a Lévy process with $\Psi(\xi) = \|\xi\|^{\alpha+o(1)}$ as $\|\xi\| \to \infty$. Then, prove that $\overline{\dim}_{_\mathrm{M}} X([0\,,1])$ and $\dim_{_\mathrm{H}} X([0\,,1])$ are both $d \wedge \alpha$ a.s.