Lecture 2

The Range of a Lévy Process

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Recap

1. Suppose $\exists C > 0$ such that $\forall \epsilon > 0$ $\exists$ and b-adic cubes $F_1, F_2, \ldots$ with $\text{diam} F_j \le \epsilon$ and $F \subseteq \cup_{j=1}^{\infty} F_j$ such that $\sum_{j=1}^{\infty} |\text{diam} F_j|^s \le C$. Then $\dim_{\text{H}} F \le s$.

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2. Suppose $\exists \mu \in \mathscr{P}(F)$ such that $I_s(\mu) < \infty$, where

$$I_s(\mu) := \iint \frac{\mu(dx)\,\mu(dy)}{|x - y|^s}.$$

   Then $\dim_{_\text{H}} F \ge s$.

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- Find a formula for $\dim_{\rm H} X([0\,,1])$, $\dim_{\rm H} X(\mathbf{R}_+)$, $\overline{\dim}_{\rm M} X([0\,,1])$, etc., where X is a Lévy process.

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  3. We will handles these matters in reverse order.

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- U is a Borel probability measure on $\mathbf{R}^d$.

A hitting bound

For all $a \in \mathbf{R}^d$ and $\epsilon > 0$ define

$$B(a\,,\epsilon) := \bigcap_{j=1}^{d} \left\{ x \in \mathbf{R}^d : \ a_j - \epsilon \le x_j < a_j + \epsilon \right\}.$$

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Lemma

For all $a \in \mathbf{R}^d$ and $\epsilon > 0$,

$$\mathrm{P}\left\{ X(s) \in B(a\,,\epsilon) \text{ for some } s \le \zeta \right\} \le \frac{U(B(a\,,2\epsilon))}{U(B(0\,,\epsilon))}.$$

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- $|X(T)-a| \le \epsilon$ a.s. on $\{T<\infty\}$.
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- Now $\sum_{B(a,2^{-n})\in\mathscr{D}_n} U(B(a\,,2^{-n})) = 1$.
- What about $\sum_{B(a,2^{-n})\in\mathscr{D}_n} U(B(a\,,2^{-n+1}))$?

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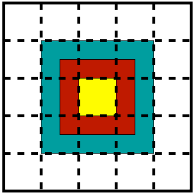

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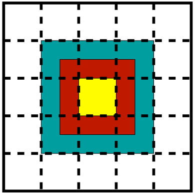

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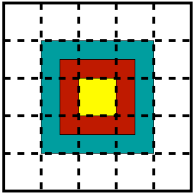

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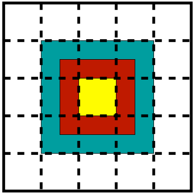

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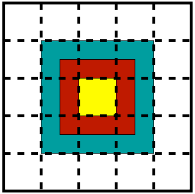

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- $\sum_{B(a,2^{-n})\in\mathscr{D}_n} U(B(a\,,2^{-n+1})) \leq 5^d \sum_{I\in\mathscr{D}_n} U(I) = 5^d$.

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This will give an upper bound for $\dim_{_\mathrm{H}} X([0\,,1])$, and another upper bound for $\overline{\dim}_{_\mathrm{M}} X([0\,,1])$.

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  2. $\dim_{\mathrm{H}} X([0,1]) \le \underline{\mathrm{ind}}U$ a.s.
- Both bounds are sharp.

  [$\dim_{\mathrm{H}}$ Pruitt, 1969; $\overline{\dim}_{\mathrm{M}}$ Taylor, XXXX]

Theorems of Pruitt and Taylor

Theorem

A.s.: $\dim_{\text{H}} X([0\,,1]) = \underline{\text{ind}}U$ *and* $\overline{\dim}_{\text{M}} X([0\,,1]) = \overline{\text{ind}}U.$

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- Let $\mu(A) := \int_0^1 \mathbf{1}_A(X(s))\, ds$ [Occupation measure]
- Strategy: $I_s(\mu) = \iint |x-y|^{-s}\, \mu(dx)\, \mu(dy) < \infty$.

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- If $s < \underline{\mathrm{ind}} U$, then $\int_0^1 \mathrm{P}\{|X(u)| \le \epsilon\} \, du = O(\epsilon^s)$. Therefore, for all $0 < s < t < \underline{\mathrm{ind}} U$,

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Need an analogue of Frostman's theorem.

Theorem (Hu and Taylor)

Suppose $F \subset \mathbf{R}^d$ is bounded measurable, and $\exists$ probability measure μ on F and $s > 0$ such that

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Therefore, $N_n(F) \ge c 2^{ns}$ i.o. ∎

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- As μ is a probab. meas. on $X([0\,,1])$, the previous theorem implies that $\overline{\dim}_{\mathrm{M}} X([0\,,1]) \geq \overline{\mathrm{ind}}\, U$ a.s. ■

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- $\therefore\ \overline{\mathrm{ind}}U = \underline{\mathrm{ind}}U = \min(d\,,2)$.

Problems

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   [Volume doubling]