Lecture 1

Measure and Dimension

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The Minkowski dimension

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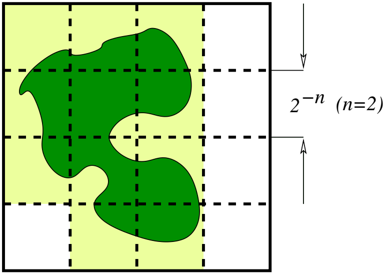

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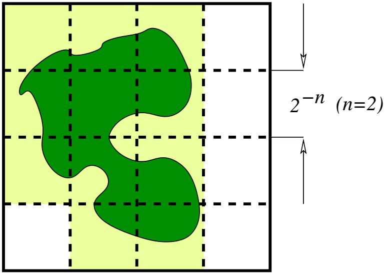
where $1 \le j_1, \ldots, j_d \le b^n$.



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where $1 \le j_1, \ldots, j_d \le b^n$.
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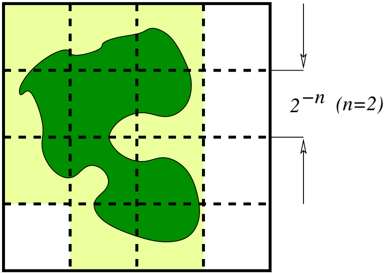

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A much better example

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- That is, $N_n(F) = b^{o(n)+n\dim_{\rm M} F}$ as $n \uparrow \infty$.

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   [covering argument].
4. Negative: $\exists$ countable sets F with $\overline{\dim}_{\mathrm{M}} F > 0$.

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- $\mathscr{H}_d\big|_{\mathscr{B}(\mathbf{R}^d)} = c\times$ *Lebesgue measure on $\mathbf{R}^d$.*

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3. $\mathscr{H}_{d+\delta}(F) = 0 \;\Rightarrow\; \dim_{\mathrm{H}} F \in [0\,, d]$.
4. *(σ-regularity)* $\dim_{\mathrm{H}} \cup_{j=1}^{\infty} F_j = \sup_{j \geq 1} \dim_{\mathrm{H}} F_j$.

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  [E.g., for the ℓ^∞ metric on $\mathbf{R}^d$, $c = 1$: An open set of diam $\le b^{-n}$ can be covered by at most 2^d b-adic cubes of diam b^{-n}]
- $\therefore \dim_{_\mathrm{H}} F$ is also equal to
  $\sup\{s > 0 : \mathscr{N}_s^b(F) > 0\} = \inf\{s > 0 : \mathscr{N}_s^b(F) < \infty\}$,
  for any and all integers $b \ge 2$.

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- Thus, we obtain $\dim_{\mathrm{H}} C \leq \log_3 2 \approx 0.6309$.
- We will prove later that this is an equality [Hausdorff].

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- Let $F_j := [j/n\,,(j+1)/n]$ for $j = 0, \ldots, n-1$. Then, with probab. one, $\text{diam}\ W(F_j) \le V_\eta n^{-\frac12 + \eta} := \epsilon \ \ \forall j$.

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- $\mathscr{H}_s^{\epsilon}(W([0\,,1])) \le V_\eta n^{1-\frac{s}{2}+s\eta} \ \Rightarrow \ \mathscr{H}_s(W([0\,,1])) < \infty$ a.s. for $s = (\frac12 - \eta)^{-1}$.

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- $\mathscr{H}_s^{\epsilon}(W([0\,,1])) \le V_\eta n^{1-\frac{s}{2}+s\eta} \;\Rightarrow\; \mathscr{H}_s(W([0\,,1])) < \infty$ a.s. for $s = (\frac12 - \eta)^{-1}$.
- $\therefore\; \dim_{\mathrm{H}} W([0\,,1]) \le 2$ a.s. We are done because $W([0\,,1]) \subset \mathbf{R}^d$. ∎

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Let F be a bounded meas. subset of $\mathbf{R}^d$. Suppose there exists $s > 0$ and $\mu \in \mathscr{P}(F)$ such that

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- $I_s(\mu) :=$ the s-dimensional [Bessel-] Riesz energy of μ.

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Simply take $Z = |\mathrm{diam}(F_J)|^s / \mu(F_J)$, where $\mathrm{P}\{J = j\} = \mu(F_j)$.

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$\mathscr{N}_s^2(F) \geq 1/I_s(\mu) > 0$. ■

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- $\mu(A) := \mathrm{P}\{X \in A\}$.
  Cantor–Lebesgue measure
  We will prove that $I_s(\mu) < \infty$ for all $s \in (0\,,\log_3 2)$.

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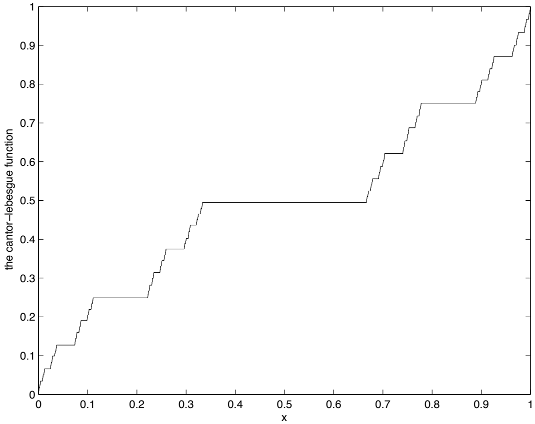
where $N := \min\{j : X_j \neq Y_j\}$. Therefore, $I_s(\mu) \le \mathrm{E}[3^{Ns}]$.
- $\mathrm{P}\{N = k\} = 2^{-k}$ for $k \ge 1$ [geometric distribution].
- $\mathrm{E}[3^{Ns}] = \sum_{k=1}^{\infty} 3^{ks} 2^{-k} < \infty$ iff $s < \log_3 2$. ■

The Cantor–Lebesgue function

$c(x) := \mu([0\,,x]) \Rightarrow c'(x) = 0$ a.e., $c(0) = 0$, $c(1) = 1$, $c =$ continuous.



"The devil's staircase" [Mandelbrot]

Brownian motion

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- Need a probability measure on $W([0\,,1])$ such that $I_s(\mu) < \infty$ a.s. for $s < \min(d\,,2)$.

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- $s < d \wedge 2 \Rightarrow I_s(\mu) \overset{\text{a.s.}}{<} \infty \Rightarrow \dim_{_H} W([0\,,1]) \overset{\text{a.s.}}{\geq} d \wedge 2$. ■

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More advanced problems

$f : [0\,,1] \to \mathbf{R}^d$ is *Hölder continuous with index* $\alpha > 0$ if

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3. Prove that $C - C := \{x - y : x, y \in C\} = [-1, 1]$. (Hint: $x \in C$, $t \in [-1, 1] \Rightarrow$ the line $y = x + t$ intersects $C \times C$ at some 3-adic square.)