# HANDOUT ON RIESZ KERNELS AND FOURIER ANALYSIS

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## 1. The result

The goal of this note is to prove the following classical theorem. Throughout,

$$\kappa_s(x) := \|x\|^{-s} \qquad \text{for all } x \in \mathbf{R}^d. \tag{1}$$

**Theorem 1.1.** *If $\alpha \in (0\,, d)$, then the $L^2$-Fourier transform of $\kappa_\alpha$ is $c\kappa_{d-\alpha}$ for a constant $c = c_{d,\alpha} \in (0\,,\infty)$.*

*Remark* 1.2. $\kappa_\alpha$ is not in $L^1$. Therefore, the Fourier transform is in the sense of $L^2$: For all rapidly-decreasing Schwartz functions $\varphi : \mathbf{R}^d \to \mathbf{R}$,

$$\int_{\mathbf{R}^d} \varphi(x)\kappa_\alpha(x)\,dx = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \overline{\hat{\varphi}(\xi)}\ \hat{\kappa}_{d-\alpha}(\xi)\,d\xi. \tag{2}$$

*Proof.* We begin with the following identity: If $\beta > -1$ and $\theta > 0$, then

$$\int_0^\infty e^{-t\|\xi\|^2} t^\beta\,dt = \frac{\Gamma(1+\beta)}{\|\xi\|^{2+2\beta}}. \tag{3}$$

Therefore, consider an arbitrary function $\varphi : \mathbf{R}^d \to \mathbf{R}$ such that $\varphi$ and its Fourier transform decay faster than any polynomial at infinity. [This defines the class $\mathscr{S}(\mathbf{R}^d)$ of rapidly-decreasing functions of L. Schwartz, and is dense in $C(\mathbf{R}^d)$.] Then, we apply the preceding with $\beta := (\alpha-2)/2$ to find that

$$\int_{\mathbf{R}^d} \frac{\varphi(\xi)}{\|\xi\|^\alpha}\,d\xi = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{(\alpha-2)/2}\left(\int_{\mathbf{R}^d}\varphi(\xi)\ e^{-t\|\xi\|^2}\,d\xi\right)dt. \tag{4}$$

We apply the Parseval identity to the middle integral to obtain

$$\left(\int_{\mathbf{R}^d}\varphi(\xi)\ e^{-t\|\xi\|^2}\,d\xi\right) = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}\overline{\hat{\varphi}(\xi)}\ \frac{e^{-\|\xi\|^2/(2t)}}{(2\pi t)^{d/2}}\,d\xi. \tag{5}$$

We plug this back into (4) to find that

$$\int_{\mathbf{R}^d} \frac{\varphi(\xi)}{\|\xi\|^\alpha}\,d\xi = c_1\int_0^\infty t^{(\alpha-2)/2}\left(\int_{\mathbf{R}^d}\overline{\hat{\varphi}(\xi)}\ \frac{e^{-\|\xi\|^2/(2t)}}{(2\pi t)^{d/2}}\,d\xi\right)dt. \tag{6}$$

The decay properties of $\varphi$ and the gaussian allow us to interchange the integrals once again. Thus, we find that

$$\int_{\mathbf{R}^d} \frac{\varphi(\xi)}{\|\xi\|^\alpha}\, d\xi = c_1 \int_{\mathbf{R}^d} \overline{\hat{\varphi}(\xi)} \left( \int_0^\infty t^{(\alpha-2)/2} \frac{e^{-\|\xi\|^2/(2t)}}{(2\pi t)^{d/2}}\, dt \right) d\xi. \tag{7}$$

A change of variables shows that

$$\int_0^\infty t^{(\alpha-2)/2} \frac{e^{-\|\xi\|^2/(2t)}}{(2\pi t)^{d/2}}\, dt = \|\xi\|^{\alpha-d} \int_0^\infty s^{(\alpha-2)/2} \frac{e^{-1/(2s)}}{(2\pi s)^{d/2}}\, ds, \tag{8}$$

which has the form $c_2\kappa_{d-\alpha}(\xi)$. Since $\alpha \in (0\,, d)$, the last integral is finite. Let $c_{d,\alpha} := (2\pi)^d c_1 c_2$ to deduce (2) and hence the theorem. □