Dynamical Processes

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- Parallel relations to noise sensitivity (Kalai et al.)

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- $t \mapsto (X_1(t)\,, X_2(t)\,, X_3(t)\,, \ldots)$ is a cadlag strong Markov process with invariant meas. μ^∞

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- Unusual times (Benjamini, Häggström, Peres, and Steif, 2003):

$$\mathrm{P}\left\{ {}^{\exists}t \geq 0 : \ R_n(t) \geq a_n \text{ i.o.} \right\} = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} a_n^{1} p^{a_n} < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} a_n^{1} p^{a_n} = \infty. \end{cases}$$

4. Runs with Contaminations

- For integer $\ell \geq 0$ fixed, let $R_n^\ell(t)$ denote the largest $j \geq \ell + 1$ such that $X_k(t) = 1$ for all but ℓ values of $k \in \{n, \ldots, n + j - 1\}$

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- Borel–Cantelli lemma exercise: For $t \geq 0$ fixed,

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- Kh., Levin, and Méndez (2007):

$$\mathrm{P}\left\{{}^\exists t \geq 0 : \; R_n^\ell(t) \geq a_n \text{ i.o.}\right\} = \begin{cases} 0 & \text{if } \sum_{n=1}^\infty a_n^{1+\ell} p^{a_n} < \infty, \\ 1 & \text{if } \sum_{n=1}^\infty a_n^{1+\ell} p^{a_n} = \infty. \end{cases}$$

Verifies a conjecture of Révész (2005).

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- Use: Stationarity; covering; asymp. independence

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- Kh., Levin, and Méndez: OK for all s, sharp (different) integral tests, etc.

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- $\rho \leftrightarrow \infty$ means there exists an infinite ray of all ones, starting with ρ
- Lyons (1990; 1992) proved that $\mathrm{P}_p\{\rho \leftrightarrow \infty\} > 0$ iff $\exists$ p.m. μ on ∂T such that

$$\iint \frac{\mu(d\sigma)\,\mu(d\tau)}{p^{|\sigma\wedge\tau|}} < \infty.$$

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  $$\mathrm{P}_p\left\{{}^{\forall}t \geq 0 : \ \rho \overset{t}{\leftrightarrow} \infty\right\} > 0$$
  - $p = p_c$? Could go either way: If $T :=$ sph. symmetric, then
  $$\mathrm{P}_p\left\{{}^{\exists}t \geq 0 : \ \rho \overset{t}{\leftrightarrow} \infty\right\} > 0 \quad \text{iff} \quad \sum_{k=1}^{\infty} \frac{p^{-k}}{k|\partial T_k|} < \infty$$

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$$\mathrm{Cap}_h(\partial T \times D) := \left[\inf_{\mu\in M_1(\partial T\times D)} I_h(\mu)\right]^{-1}$$

where

$$I_h(\mu) := \iint h((\sigma\,,s)\,;(\tau\,,t))\,\mu(d\sigma\,ds)\,\mu(d\tau\,dt).$$

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- If either D is nice, or T is nice, then $\mathrm{Cap}_h(\partial T \times D)$ can be computed explicitly in terms of the geometry of $\partial T \times D$