Lecture 4

Heat Flow

1. The Ornstein–Uhlenbeck Operator

The Laplacian $\Delta := D \cdot D := \sum_{i=1}^{n} \partial^2_{i,i}$ is one of the central differential operators in the analysis of Lebesgue spaces. In other words, Δ is the dot product of D with the negative of its adjoint. The analogue of the Laplacian in Gauss space is the generalized differential operator

$$\mathscr{L} := -A \cdot D := -\sum_{j=1}^{n} A_j D_j \tag{4.1}$$

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which is called the *Ornstein–Uhlenbeck operator* on the Gauss space. We can think of $\mathscr{L}$ in the form, $(\mathscr{L} g)(x) = \sum_{i=1}^{n} (D^2_{i,i} g)(x) - \sum_{i=1}^{n} x_i (D_i g)(x)$, or as random variables as

$$\mathscr{L} g = \sum_{i=1}^{n} D^2_{i,i} g - Z \cdot (Dg) = \sum_{i=1}^{n} D^2_{i,i} g - Z \cdot (Dg)(Z). \tag{4.2}$$

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The preceding makes sense as an identity in $L^2(\mathbb{P}_n)$ whenever $g \in \mathbb{D}^{2,2}(\mathbb{P}_n)$ and $Z_i(D_i g)(Z)$ is in $L^2(\mathbb{P}_n)$ for every $1 \leqslant i \leqslant n$. And when $g \in C^2(\mathbb{R})$, then

$$(\mathscr{L} g)(x) = (\Delta g)(x) - x \cdot (\nabla g)(x),$$

for every $x \in \mathbb{R}$.

Definition 1.1. The *domain of the definition* of $\mathscr{L}$ is

$$\mathrm{Dom}[\mathscr{L}] := \left\{ g \in \mathbb{D}^{2,2}(\mathbb{P}_n) : Z \cdot (Dg)(Z) \in L^2(\mathbb{P}_n) \right\}.$$

If $g \in \mathbb{D}^{2,2}(\mathbb{P}_n)$, then

$$\mathbb{E}\left(|Z \cdot (Dg)(Z)|^2\right) \leqslant \mathbb{E}\left(\|Z\|^2\right)\mathbb{E}\left(\|Dg\|^2\right) \leqslant n\|g\|_{2,2}^2. \tag{4.3}$$ Dom:L:n

Therefore, we see immediately that

$$\mathrm{Dom}[\mathcal{L}] = \mathbb{D}^{2,2}(\mathbb{P}_n).$$

I will sometimes emphasize the domain of $\mathcal{L}$ by writing it as $\mathrm{Dom}[\mathcal{L}]$ rather than $\mathbb{D}^{2,2}(\mathbb{P}_n)$, since in infinite dimensions the domain of $\mathcal{L}$ is frequently not all of $\mathbb{D}^{2,2}(\mathbb{P}_n)$. Technically, this assertion manifests itself in (4.3) via the appearance of the multiplicative factor n, which becomes infinitely large in infinite dimensions.

It is not difficult to see how $\mathcal{L}$ acts on Hermite polynomials. The following hashes out the details of that computation.

lem:L:H **Lemma 1.2.** *$\mathcal{L}\mathcal{H}_k = -|k|\mathcal{H}_k$ for every $k \in \mathbb{Z}_+^n$, where $|k| := \sum_{i=1}^n k_i$.*

Proof. We apply (3.2) [p. 33] to see that $A_jD_j\mathcal{H}_k = k_j\mathcal{H}_k$ for all $k \in \mathbb{Z}_+^n$ and $1 \leqslant j \leqslant n$. Sum over j to finish. □

In other words, for every $k \in \mathbb{Z}_+^n$, the Hermite polynomial $\mathcal{H}_k$ is an eigenfunction of $\mathcal{L}$, with eigenvalue $-|k|$. Since

$$f = \sum_{k\in\mathbb{Z}_+^n} \frac{\mathbb{E}(f\mathcal{H}_k)}{k!}\mathcal{H}_k \qquad \text{in } L^2(\mathbb{P}_n),$$

it follows readily that

$$\mathcal{L}f = -\sum_{k\in\mathbb{Z}_+^n} \frac{|k|}{k!}\,\mathbb{E}(f\mathcal{H}_k)\mathcal{H}_k \qquad \text{in } L^2(\mathbb{P}_n), \tag{4.4}$$ L

whence

$$\mathrm{Dom}[\mathcal{L}] = \mathbb{D}^{2,2}(\mathbb{P}_n) = \left\{ f \in L^2(\mathbb{P}_n): \sum_{k\in\mathbb{Z}_+^n} \frac{|k|^2}{k!}\left|\mathbb{E}(f\mathcal{H}_k)\right|^2 < \infty \right\}. \tag{4.5}$$ Dom:L

Define, for every $t \geqslant 0$,

$$P_tf := P(t)f := \sum_{k\in\mathbb{Z}_+^n} \frac{\mathrm{e}^{-|k|t}}{k!}\,\mathbb{E}(f\mathcal{H}_k)\mathcal{H}_k, \tag{4.6}$$ P(t)

where the identity holds in $L^2(\mathbb{P}_n)$.

pr:heat **Proposition 1.3.** *If $f \in \mathrm{Dom}[\mathcal{L}]$, then $P_tf \in \mathrm{Dom}[\mathcal{L}]$ for all $t > 0$. Moreover, $u(t) := P_tf$ is the unique $L^2(\mathbb{P}_n)$-valued solution to the generalized* *partial differential equation,*

$$\begin{cases} \dfrac{\partial}{\partial t}u(t) = \mathscr{L}[u(t)], \quad \text{for all } t > 0, \text{ subject to} \\ u(0) = f. \end{cases} \tag{4.7}$$

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Definition 1.4. The family $\{P_t\}_{t\geqslant 0}$ is called the *Ornstein–Uhlenbeck semigroup*, and the linear partial differential equation (4.7) is the *heat equation for the Ornstein–Uhlenbeck operator* $\mathscr{L}$.

Proof. By (4.6),

$$\mathbb{E}\left[P_t(f)\mathscr{H}_k\right] = \mathrm{e}^{-|k|t}\,\mathbb{E}\left[f\mathscr{H}_k\right] \qquad \text{for all } t \geqslant 0 \text{ and } k \in \mathbb{Z}^n_+.$$

Therefore,

$$\sum_{k\in\mathbb{Z}^n_+} \frac{|k|}{k!}\left|\mathbb{E}\left[P_t(f)\mathscr{H}_k\right]\right|^2 = \sum_{k\in\mathbb{Z}^n_+} \frac{|k|\mathrm{e}^{-|k|t}}{k!}\left|\mathbb{E}\left[f\mathscr{H}_k\right]\right|^2 \leqslant \sum_{k\in\mathbb{Z}^n_+} \frac{|k|}{k!}\left|\mathbb{E}\left[f\mathscr{H}_k\right]\right|^2$$

is finite. This proves that $P_tf \in \mathrm{Dom}[\mathscr{L}]$ for all $t > 0$.

It is intuitively clear from (4.4) and (4.6) that $\partial P_tf/\partial t = \mathscr{L}f$, when u solves (4.7). But since P_tf and $\mathscr{L}f$ are not numbers, rather elements of $L^2(\mathbb{P}_n)$, let us write the details: We know that $u(t) \in L^2(\mathbb{P}_n)$ for every $t \geqslant 0$, and that

$$\mathbb{E}[gu(t)] = \sum_{k\in\mathbb{Z}^n_+} \frac{\mathrm{e}^{-|k|t}}{k!}\,\mathbb{E}\left[f\mathscr{H}_k\right]\mathbb{E}\left[g\mathscr{H}_k\right],$$

for all $t \geqslant 0$ and $g \in L^2(\mathbb{P}_n)$. It is not hard to see that the time derivative operator commutes with the sum to yield

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}[gu(t)] &= -\sum_{k\in\mathbb{Z}^n_+} \frac{|k|\mathrm{e}^{-|k|t}}{k!}\,\mathbb{E}\left[f\mathscr{H}_k\right]\mathbb{E}\left[g\mathscr{H}_k\right] \\ &= \mathbb{E}\left[g\,\mathscr{L}[u(t)]\right] \qquad \text{for all } t \geqslant 0, \end{aligned}$$

since $\mathbb{E}(\mathscr{H}_k\,\mathscr{L}[u(t)]) = -|k|(k!)^{-1}\,\mathbb{E}[u(t)\mathscr{H}_k]$ for all $k \geqslant 0$, by (4.4). Thus, u solves the PDE (4.7).

If v is another $L^2(\mathbb{P}_n)$-valued solution to (4.7), then $\phi := u - v$ solves

$$\begin{cases} \dfrac{\partial}{\partial t}\phi(t) = \mathscr{L}[\phi(t)], \quad \text{subject to} \\ \phi(0) = 0. \end{cases}$$

Project ϕ on to $\mathscr{H}_k$, where $k \in \mathbb{Z}^n_+$ is fixed, in order to find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\left[\phi(t)\mathscr{H}_k\right] = -|k|\,\mathbb{E}\left[\phi(t)\mathscr{H}_k\right],$$

by (4.4). Since $\mathbb{E}[\phi(0)\mathscr{H}_k] = 0$, it follows that $\mathbb{E}[\phi(t)\mathscr{H}_k] = 0$ for all $t \geqslant 0$ and $k \in \mathbb{Z}^n_+$. The completeness of the Hermite polynomials [Theorem

2.1] ensures that $\phi(t) = 0$ for all $t \geqslant 0$. This implies the remaining uniqueness portion of the proposition. □

OU:Semigrp

Proposition 1.5. *The family* $\{P_t\}_{t\geqslant 0}$ *is a* symmetric Markov semigroup *on* $L^2(\mathbb{P}_n)$. *That is:*

(1) *Each* P_t *is a linear operator from* $L^2(\mathbb{P}_n)$ *to* $L^2(\mathbb{P}_n)$*, and* $P_t1 = 1$*;*
(2) $P_0 :=$ *the identity map. That is,* $P_0 f = f$ *for all* $f \in L^2(\mathbb{P}_n)$*;*
(3) *Each* P_t *is self-adjoint. That is,*

$$\mathbb{E}[gP_t(f)] = \mathbb{E}[P_t(g)f] \qquad \text{for all } f, g \in L^2(\mathbb{P}_n) \text{ and } t \geqslant 0;$$

(4) *Each* $P_t : L^2(\mathbb{P}_n) \to L^2(\mathbb{P}_n)$ *is non expansive. That is,*

$$\mathbb{E}(|P_t f|^2) \leqslant \mathbb{E}(|f|^2) \qquad \text{for all } f \in L^2(\mathbb{P}_n) \text{ and } t \geqslant 0;$$

(5) $\{P_t\}_{\geqslant 0}$ *is a semigroup of linear operators. That is,*

$$P_{t+s} = P_t P_s = P_s P_t \qquad \text{for all } s, t \geqslant 0.$$

Finally, $\mathbb{P}_n$ *is* invariant *for* $\{P_t\}_{t\geqslant 0}$. *That is,*

$$\mathbb{E}[P_t f] = \int P_t f \, \mathrm{d}\mathbb{P}_n = \int f \, \mathrm{d}\mathbb{P}_n = \mathbb{E}(f) = \lim_{s\uparrow\infty} P_s f \qquad \text{a.s. and in } L^2(\mathbb{P}_n).$$

Proof. Parts (1) and (2) are immediate consequences of the definition (4.6) of P_t. [For example, $P_t 1 = 1$ because $\mathscr{H}_0 = 1$.]

Part (3) follows since

$$\mathbb{E}[gP_t f] = \sum_{k\in\mathbb{Z}^n_+} \frac{\mathrm{e}^{-|k|t}}{k!} \mathbb{E}[f\mathscr{H}_k]\, \mathbb{E}[g\mathscr{H}_k],$$

which is clearly a symmetric form in (f, g). Part (4) is a consequence of the following calculation.

$$\mathbb{E}\left(|P_t f|^2\right) = \sum_{k\in\mathbb{Z}^n_+} \frac{\mathrm{e}^{-2|k|t}}{k!} |\mathbb{E}[f\mathscr{H}_k]|^2 \leqslant \sum_{k\in\mathbb{Z}^n_+} \frac{1}{k!} |\mathbb{E}[f\mathscr{H}_k]|^2 = \mathbb{E}\left(|f|^2\right).$$

Finally, we observe that $\mathbb{E}[P_s f\mathscr{H}_k] = \mathrm{e}^{-|k|s}/(k!)\,\mathbb{E}[f\mathscr{H}_k]$ for all real numbers $s \geqslant 0$ and integral vectors $k \in \mathbb{Z}^n_+$. Therefore,

$$P_t[P_s f] = \sum_{k\in\mathbb{Z}^n_+} \frac{\mathrm{e}^{-|k|t}}{k!} \mathbb{E}[P_s(f)\mathscr{H}_k]\,\mathscr{H}_k = P_{t+s} f.$$

Since $P_{t+s} = P_{s+t}$, this shows also that $P_s P_t = P_t P_s$, and verifies (5).

In order to finish the proof we need to verify the invariance of $\mathbb{P}_n$. First of all note that $1(x) := 1$ is in $L^2(\mathbb{P}_n)$. Therefore,

$$\mathbb{E}[P_t f] = \sum_{k\in\mathbb{Z}^n_+} \frac{\mathrm{e}^{-|k|t}}{k!} \mathbb{E}[f\mathscr{H}_k]\, \mathbb{E}(\mathscr{H}_k),$$

which is equal to $\mathbb{E}(f\mathcal{H}_0) = \mathbb{E}(f)$ since $\mathbb{E}(\mathcal{H}_k) = \mathbb{E}(\mathcal{H}_0\mathcal{H}_k) = 0$ for all $k \in \mathbb{Z}^n_+ \setminus \{0\}$ [Theorem 2.1]. By (4.6),

$$P_t f = \sum_{k\in\mathbb{Z}^n_+} \frac{\mathrm{e}^{-|k|t}}{k!}\, \mathbb{E}\left[f\mathcal{H}_k\right] \mathcal{H}_k \qquad \text{a.s.,} \tag{4.8}$$

P(t)f:H

where the convergence holds in $L^2(\mathbb{P}_n)$.

The Cauchy–Schwarz inequality yields $|\langle f\,, \mathcal{H}_k\rangle_{L^2(\mathbb{P}_n)}| \leqslant \|f\|_{L^2(\mathbb{P}_n)}$, valid for all $k \in \mathbb{Z}^n_+$. Therefore, the identity $\|\mathcal{H}_k\|_{L^2(\mathbb{P}_n)} = 1$ and the Minkowski inequality together imply that

$$\left\| \sup_{t\geqslant 0} \sum_{k\in\mathbb{Z}^n_+} \frac{\mathrm{e}^{-|k|t}}{k!} \left|\mathbb{E}\left[f\mathcal{H}_k\right] \mathcal{H}_k\right| \right\|_{L^2(\mathbb{P}_n)} \leqslant \|f\|_{L^2(\mathbb{P}_n)} \sum_{k\in\mathbb{Z}^n_+} \frac{1}{k!} < \infty.$$

In particular, the sum in (4.8) also converges absolutely, uniformly in $t \geqslant 0$, with $\mathbb{P}_n$-probability one. Consequently,

$$\lim_{t\uparrow\infty} P_t f = \sum_{k\in\mathbb{Z}^n_+} \lim_{t\to\infty} \frac{\mathrm{e}^{-|k|t}}{k!}\, \mathbb{E}\left[f\mathcal{H}_k\right] \mathcal{H}_k = \mathbb{E}\left[f\mathcal{H}_0\right],$$

almost surely. The final quantity is equal to $\mathbb{E}(f)$, as desired. □

2. Mehler's Formula

The heat equation (4.7) for the OU operator $\mathcal{L}$ is just the initial-value problem,

$$\left[\begin{array}{ll} \dfrac{\partial}{\partial t} u(t\,,x) = (\Delta u)(t\,,x) + x\cdot(\nabla u)(t\,,x) & [t > 0,\ x \in \mathbb{R}^n], \\ u(0\,,x) = f(x) & [x \in \mathbb{R}^n], \end{array}\right.$$

but written out in an infinite-dimensional manner. As such, it can be solved by other, more elementary, methods as well. We have taken this route in order to introduce the OU semigroup $\{P_t\}_{t\geqslant 0}$ and the associated OU operator $\mathcal{L}$. These objects will play a central role in Gaussian analysis, more so than does the heat equation itself. Still, it might be good to know that every $L^2(\mathbb{P}_n)$-valued solution is also a classical solution when, for example, f is in $C^2_0(\mathbb{P}_n)$. Among many other things, this fact follows immediately from the following interesting formula and the dominated convergence theorem.

Mehler

Theorem 2.1 (Mehler's Formula). *If $f \in L^2(\mathbb{P}_n)$ and $t \geqslant 0$, then*

$$(P_t f)(x) = \mathbb{E}\left[f\left(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}\,Z\right)\right],$$

for almost every $x \in \mathbb{R}$.

rem:Mehler **Remark 2.2.** One of the many by-products of Mehler's formula is the fact that each mapping $f \mapsto P_t f$ is a Bochner integral; in particular, every P_t satisfies the "Cauchy–Schwarz inequality," which is stronger than the non-expansiveness of P_t:

$$|P_t f|^2 \leqslant P_t(f^2) \qquad \text{a.s. for all } t \geqslant 0 \text{ and } f \in L^2(\mathbb{P}_n).$$

Proof. I will first prove the result for $n = 1$ since the notation is simpler in that case.

By density it suffices to prove the result for all $f \in C_0^\infty(\mathbb{P}_1)$. Define for such functions f and $t \geqslant 0$,

$$(T_t f)(x) := \mathbb{E}\left[f\left(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}\,Z\right)\right],$$

for every $x \in \mathbb{R}$. Both sides are C^∞ functions in either variable t and x [dominated convergence]. Our goal is to prove that $T_t = P_t f$ for all $t \geqslant 0$. This will complete the proof. Note that

$$\begin{aligned}(T_t f)(x) &= \int_{\mathbb{R}^n} f\left(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}\,z\right)\gamma_n(z)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^n} f(y)\gamma_n\left(\frac{y-\mathrm{e}^{-t}x}{\sqrt{1-\mathrm{e}^{-2t}}}\right)\mathrm{d}y.\end{aligned}$$

Since $f \in C_0^\infty(\mathbb{P}_n)$, we can differentiate under the integral any number of times we want in order to see that $\partial(T_t f)/\partial t = \mathscr{L}(T_t f)$, after a few lines of calculus applied to the function γ_n. Since $T_0 f = f$, the uniqueness portion of Proposition 1.3 implies that $T_t f = P_t f$ for all $t \geqslant 0$. □

3. A Covariance Formula

One of the highlights of our analysis so far is that it leads to an explicit formula for $\mathrm{Cov}(f\,, g)$ for a large number of nice functions f and g. Before we discuss that formula, let us observe the following.

lem:DP:PD **Lemma 3.1.** *For all $t \geqslant 0$ and $1 \leqslant j \leqslant n$,*

$$D_j P_t = \mathrm{e}^{-t} P_t D_j \qquad \textit{and} \qquad A_j P_t = P_t A_j.$$

Consequently, $\mathscr{L}(P_t f) = \exp(-t) P_t(\mathscr{L} f)$, also.

Proof. First consider the case that $n = 1$. In that case,

$$P_t f = \sum_{k=0}^\infty \frac{\mathrm{e}^{-kt}}{k!}\,\mathbb{E}(fH_k)H_k,$$

for all $f \in L^2(\mathbb{P}_1)$. Therefore, whenever $f \in \mathbb{D}^{1,2}(\mathbb{P}_1)$,

$$\begin{aligned} DP_t f &= \sum_{k=0}^{\infty} \frac{e^{-kt}}{k!}\,\mathbb{E}(fH_k)DH_k = \sum_{k=0}^{\infty} \frac{ke^{-kt}}{k!}\,\mathbb{E}(fH_k)H_{k-1} \\ &= e^{-t}\sum_{k=0}^{\infty} \frac{e^{-kt}}{k!}\,\mathbb{E}\left[fH_{k+1}\right]H_k, \end{aligned} \tag{4.9}$$

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by (3.2) [page 33]. Similarly, $\mathbb{E}[D(f)H_k] = \mathbb{E}[fA(H_k)] = \mathbb{E}[fH_{k+1}]$ for all $k \geqslant 0$. Therefore,

$$P_t Df = \sum_{k=0}^{\infty} \frac{e^{-kt}}{k!}\,\mathbb{E}\left[fH_{k+1}\right]H_k.$$

Match this expression with (4.9) in order to see that $DP_t = \exp(-t)P_tD$ when $n = 1$. A similar argument shows that $AP_t = P_tA$ in this case as well.

When $n \geqslant 1$ and $f = f_1 \otimes \cdots \otimes f_n$ for $f_1, \ldots, f_n \in \mathbb{D}^{1,2}(\mathbb{P}_1)$, we can check that

$$\left(D_j P_t f\right)(x) = \prod_{\substack{1\leqslant q\leqslant n \\ q\neq j}} \left(P_t^{(q)} f_q\right)(x_q) \times (D_j P_t^{(j)} f_j)(x_j) = e^{-t}(P_t D_j f)(x),$$

by the one-dimensional part of the proof that we just developed. Since every $f \in \mathbb{D}^{1,2}(\mathbb{P}_n)$ can be approximated arbitrarily well by functions of the form $f_1 \otimes \cdots \otimes f_n$, where $f_j \in \mathbb{D}^{1,2}(\mathbb{P}_1)$, it follows that $D_jP_t = \exp(-t)P_tD_j$ on $\mathbb{D}^{1,2}(\mathbb{P}_n)$.

Similarly, one proves that $A_jP_t = P_tA_j$ in general.

To finish note that

$$\mathscr{L}P_t = -\sum_{j=1}^{n} A_jD_jP_t = -e^{-t}\sum_{j=1}^{n} A_jP_tD_j = -e^{-t}\sum_{j=1}^{n} P_tA_jD_j = e^{-t}\mathscr{L}P_t.$$

This completes the proof. □

Lemma 3.1 has the following important corollary.

pr:Cov **Proposition 3.2.** *For every $f, g \in \mathbb{D}^{1,2}(\mathbb{P}_n)$,*

$$\mathrm{Cov}(f\,,g) = \int_0^\infty e^{-t}\,\mathbb{E}\left[(Df)\cdot(P_tDg)\right]dt,$$

where $P_tDg = (P_tD_1g\,, \ldots, P_tD_ng)$.

Proof. Recall from Proposition 1.5 that $P_tg \to \mathbb{E}[g]$ in $L^2(\mathbb{P}_n)$ as $t \to \infty$, and $P_0g = g$. Therefore,

$$g(x) - \mathbb{E}[g] = -\int_0^\infty \frac{\partial}{\partial t}(P_tg)(x)\,dt,$$

where the identity is understood to hold in $L^2(\mathbb{P}_n)$, and the integral converges in $L^2(\mathbb{P}_n)$ as well. Therefore, by Fubini's theorem,

$$\begin{aligned}\mathrm{Cov}(f\,,g) = \mathbb{E}\left[f(Z)(g(Z)-\mathbb{E}[g])\right] &= -\int_0^\infty \mathbb{E}\left[f(Z)\frac{\partial}{\partial t}(P_t g)(Z)\right]\mathrm{d}t\\ &= -\int_0^\infty \mathbb{E}\left[f(Z)(\mathscr{L}P_t g)(Z)\right]\mathrm{d}t,\end{aligned}$$

since $P_t g$ solves the heat equation for the operator $\mathscr{L}$. Next we may observe that, since $\mathscr{L} = -A\cdot D$ and A_j is the adjoint to D_j,

$$\begin{aligned}\mathbb{E}\left[f(Z)(\mathscr{L}P_t g)(Z)\right] &= -\sum_{j=1}^n \mathbb{E}\left[(D_j f)(Z)(D_j P_t g)(Z)\right]\\ &= -\mathrm{e}^{-t}\sum_{j=1}^n \mathbb{E}\left[(D_j f)(Z)(P_t D_j g)(Z)\right].\end{aligned}$$

We have appealed to Lemma 3.1 in the second line. This concludes the proof. □

Let us conclude with a quick application of Proposition 3.2.

A Second Proof of the Poincaré Inequality. For every $f \in \mathbb{D}^{1,2}(\mathbb{P}_n)$ and $t \geqslant 0$,

$$\left|\mathbb{E}\left[(Df)\cdot(P_t Df)\right]\right| \leqslant \left|\mathbb{E}\left(\|Df\|^2\right)\mathbb{E}\left(\|P_t Df\|^2\right)\right|^{1/2},$$

is at most $\mathbb{E}(\|Df\|^2)$ since P_t is non-expanding on $L^2(\mathbb{P}_n)$ [Proposition 1.5]. Therefore, the Poincaré inequality follows from Proposition 3.2. □

4. The Resolvent of the Ornstein–Uhlenbeck Semigroup

The classical theory of linear semigroups tells us that it is frequently better to study a semigroup of linear operators via its "resolvent." In the present context, this leads us to the following.

Definition 4.1. The *resolvent* of the OU semigroup $\{P_t\}_{t\geqslant 0}$ is the family $\{R_\lambda\}_{\lambda>0}$ of linear operators defined via

$$(R_\lambda f)(x) := \int_0^\infty \mathrm{e}^{-\lambda t}(P_t f)(x)\,\mathrm{d}t, \tag{4.10}$$

for all bounded and measurable functions $f:\mathbb{R}^n\to\mathbb{R}$ and all $\lambda>0$.

Informally speaking, $R(\lambda) := \int_0^\infty \exp(-\lambda t)P(t)\,\mathrm{d}t$ defines the Laplace transform of the semigroup $\{P(t)\}_{t\geqslant 0}$, and knowing R should in principle be the same as knowing P. We will see soon that this is the case. But first let us define the resolvent not pointwise, as we just did, but as an element of the Hilbert space $L^2(\mathbb{P}_n)$.

According to Mehler's formula [Theorem 2.1], if f is bounded and measurable, then $P_t f$ is also; in fact, $\sup_x |(P_t f)(x)| \leqslant \sup_x |f(x)|$, whence the integral in (4.10) converges absolutely, uniformly in $x \in \mathbb{R}^n$. One can extend the domain of the definition of R_λ further by standard means. In fact, because P_t is non expensive on $L^2(\mathbb{P}_n)$ [Proposition 1.5],

$$\mathbb{E}\left(|P_t f|^2\right) \leqslant \mathbb{E}(f^2), \quad \text{whence} \quad \mathbb{E}\left(|R_\lambda f|^2\right) \leqslant \lambda^{-2}\, \mathbb{E}(f^2),$$

for all bounded functions $f \in L^2(\mathbb{P}_n)$ and every $t, \lambda > 0$. If $f \in L^2(\mathbb{P}_n)$ then we can find bounded functions $f_1, f_2, \ldots \in L^2(\mathbb{P}_n)$ such that $\mathbb{E}(|f_\ell - f|^2) \leqslant 2^{-\ell}$ for all $\ell \geqslant 1$, and hence the preceding inequality shows that

$$\mathbb{E}\left(|R_\lambda f_m - R_\lambda f_\ell|^2\right) \leqslant \lambda^{-2}\, \mathbb{E}\left(|f_m - f_\ell|^2\right) \leqslant \frac{2^{-\ell} + 2^{-m}}{\lambda^2},$$

for all $m, \ell \geqslant 1$. Therefore, $\ell \mapsto R_\lambda f_\ell$ is a Cauchy sequence in $L^2(\mathbb{P}_n)$ and hence $R_\lambda f := \lim_{\ell \to \infty} R_\lambda f_\ell$ is a well-defined limit in $L^2(\mathbb{P}_n)$. Since every P_t is non expansive on $L^2(\mathbb{P}_n)$, it follows similarly that (4.10) holds a.s. for all $f \in L^2(\mathbb{P}_n)$ and $\lambda > 0$. Let us pause and record these observations before going further.

pr:R **Proposition 4.2.** *For every $\lambda > 0$, R_λ is a bounded continuous linear map from $L^2(\mathbb{P}_n)$ to $L^2(\mathbb{P}_n)$, with operator norm $\leqslant \lambda^{-2}$. Finally,* (4.10) *holds a.s. for all $f \in L^2(\mathbb{P}_n)$ and $\lambda > 0$, and*

$$R_\lambda f = \sum_{k \in \mathbb{Z}^n_+} \frac{\mathbb{E}(f \mathcal{H}_k)}{k!(\lambda + |k|)}\, \mathcal{H}_k \qquad \text{a.s.,} \tag{4.11}$$

R:H

where the sum converges in $L^2(\mathbb{P}_n)$.

Proof. The only unproved part of the assertion is (4.11), which represents $R_\lambda f$ in terms of Hermite polynomials.

If $f \in L^2(\mathbb{P}_n)$, then $R_\lambda f \in L^2(\mathbb{P}_n)$ for all $\lambda > 0$, and Theorem 2.1 ensures that

$$R_\lambda f = \sum_{k \in \mathbb{Z}^n} \frac{1}{k!}\, \mathbb{E}\left[(R_\lambda f)\mathcal{H}_k\right] \mathcal{H}_k.$$

By Fubini's theorem, (4.10), and (4.6),

$$\mathbb{E}\left[(R_\lambda f)\mathcal{H}_k\right] = \int_0^\infty \mathrm{e}^{-\lambda t}\, \mathbb{E}\left[(P_t f)\mathcal{H}_k\right] \mathrm{d}t = \mathrm{e}^{-(|k|+\lambda)t}\, \mathbb{E}(f \mathcal{H}_k),$$

for all $k \in \mathbb{Z}^n$, $t \geqslant 0$, and $\lambda > 0$. Multiply the preceding by $\mathcal{H}_k / k!$ and sum over $k \in \mathbb{Z}^n$ to finish. □

pr:RE **Proposition 4.3** (The Resolvent Equation). *For all $f \in L^2(\mathbb{P}_n)$, and for every distinct pair $\alpha, \lambda > 0$,*

$$R_\lambda R_\alpha f = R_\alpha R_\lambda f = -\frac{R_\lambda f - R_\alpha f}{\lambda - \alpha} \qquad \text{a.s.} \tag{4.12}$$ RE

Proof. We apply the Fubini theorem and (4.10) a few times back-to-back as follows: Almost surely,

$$\begin{aligned} R_\lambda R_\alpha f &= \int_0^\infty \mathrm{e}^{-\lambda t} P_t(R_\alpha f)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t} P_t \left(\int_0^\infty \mathrm{e}^{-\alpha s}\ P_s f\,\mathrm{d}s \right) \mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}t \int_0^\infty \mathrm{e}^{-\alpha s}\,\mathrm{d}s\ P_{t+s} f = \int_0^\infty \mathrm{e}^{-(\lambda-\alpha)t}\,\mathrm{d}t \int_t^\infty \mathrm{e}^{-\alpha r}\mathrm{d}r\ P_r f \\ &= \int_0^\infty \mathrm{e}^{-\alpha r} P_r f\,\mathrm{d}r \int_0^r \mathrm{e}^{-(\lambda-\alpha)t}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\alpha r} P_r(f) \left(\frac{1 - \mathrm{e}^{-(\lambda-\alpha)r}}{\lambda - \alpha} \right) \mathrm{d}r. \end{aligned}$$

Reorganize the integral to finish. □

Eq. (4.12) is called the *resolvent equation*, and readily implies the following.

co:RE **Corollary 4.4.** *For every $\lambda > 0$, R_λ maps $L^2(\mathbb{P}_n)$ bijectively onto its range*

$$R_\lambda\left(L^2(\mathbb{P}_n)\right) := \left\{ R_\lambda f : f \in L^2(\mathbb{P}_n) \right\}.$$

The preceding range does not depend on $\lambda > 0$. Moreover, the range is dense in $L^2(\mathbb{P}_n)$; in fact, $\lim_{\lambda\to\infty} \lambda R_\lambda f = f$ in $L^2(\mathbb{P}_n)$ for every $f \in L^2(\mathbb{P}_n)$.

Proof. First, we observe that $x \mapsto (R_\lambda f)(x)$ is a.s. equal to a continuous function for all $f \in L^2(\mathbb{P}_n)$ and $\lambda > 0$. This follows from Mehler's formula [Proposition 2.1, page 45] and the dominated convergence theorem. Therefore, we can always redefine it so that $R_\lambda f$ is continuous. In particular, if $R_\lambda f = 0$ a.s. for some $\lambda > 0$, then $R_\lambda f \equiv 0$ and hence $R_\alpha f \equiv 0$ for all $\alpha > 0$ thanks to the resolvent equation. The uniqueness theorem for Laplace transforms now shows that if $R_\lambda f = 0$ a.s. for some $\lambda > 0$ then $f = 0$ a.s. By linearity we find that if $R_\lambda f = R_\lambda g$ a.s. for some $f, g \in L^2(\mathbb{P}_n)$ and $\lambda > 0$, then $f = g$ a.s. Consequently, R_λ is a one-to-one and onto map from $L^2(\mathbb{P}_n)$ to its range $R_\lambda(L^2(\mathbb{P}_n))$.

Next, let us suppose that g is the range of R_α; that is, $g = R_\alpha f$ for some $f \in L^2(\mathbb{P}_n)$. By the resolvent equation,

$$R_\lambda g = -\frac{R_\lambda f - g}{\lambda - \alpha} \quad \Rightarrow \quad g = (\lambda - \alpha) R_\lambda g - R_\lambda f = R_\lambda h,$$

for $h = (\lambda - \alpha) g - f$. This shows that g is in the range of R_λ, whence $R_\alpha(L^2(\mathbb{P}_n)) \subset R_\lambda(L^2(\mathbb{P}_n))$. Reverse the roles of α and λ to see that $R_\lambda(L^2(\mathbb{P}_n))$ does not depend on $\lambda > 0$.

Finally, we verify the density assertion. Let $f \in L^2(\mathbb{P}_n)$, and recall [Proposition 1.5, page 44] that $P_tf \to f$ in $L^2(\mathbb{P}_n)$ as $t \downarrow 0$. By this and the dominated convergence theorem,

$$\lambda R_\lambda f = \lambda \int_0^\infty \mathrm{e}^{-\lambda t} P_t f \, \mathrm{d}t = \int_0^\infty \mathrm{e}^{-s} P_{s/\lambda} f \, \mathrm{d}s \to f \qquad \text{in } L^2(\mathbb{P}_n),$$

as $\lambda \uparrow \infty$. This implies that the range of the resolvent in dense in $L^2(\mathbb{P}_n)$ because it proves that for all $\varepsilon > 0$ there exists an elements of the range $R_1(L^2(\mathbb{P}_n)) = \cup_{\alpha>0} R_\alpha(L^2(\mathbb{P}_n))$—namely $\lambda R_\lambda f = R_\lambda(\lambda f)$ for a sufficiently large λ—that is close to within ε of f in the $L^2(\mathbb{P}_n)$ norm. □

Corollary 4.4 tells us that we can in principle compute the entire semigroup $\{P_t\}_{t\geqslant 0}$ from the operator R_λ for a given $\lambda > 0$. And of course the converse is also true by (4.10). From now on we will consider $\lambda = 1$ only.

Definition 4.5. If $f \in L^2(P_n)$ then $R_1 f$ is called the *one-potential* of f. The linear operator R_1 is also known as the [one-] *potential operator*.

lem:R **Lemma 4.6.** *R_1 is a non-expansive and symmetric linear operator on $L^2(\mathbb{P}_n)$.*

Proof. Linearity is obvious. We need to prove that for all $f, g \in L^2(\mathbb{P}_n)$:

(1) $\mathbb{E}(|R_1 f|^2) \leqslant \mathbb{E}(f^2)$; and
(2) $\mathbb{E}[g(R_1 f)] = \mathbb{E}[(R_1 g) f]$.

They follow from the corresponding properties of the semigroup $\{P_t\}_{t\geqslant 0}$, and (4.10). □

The potential operator arises naturally in a number of ways. For example, Proposition 3.2 can be recast in terms of the potential operator as follows:

th:Cov:1 **Theorem 4.7** (Houdré, Pérez-Abreu, and Surgailis, XXX). *For every $f, g \in \mathbb{D}^{1,2}(\mathbb{P}_n)$,*

$$\mathrm{Cov}(f\,, g) = \mathbb{E}\left[\langle Df\,, Dg\rangle_{R_1}\right],$$

where

$$\langle p\,, q\rangle_{R_1} := p \cdot (R_1 q) \qquad \textit{for all } p, q \in L^2(\mathbb{P}_n \times \chi_n), \tag{4.13}$$ energy

and $R_1 q = (R_1 q_1\,, \ldots, R_1 q_2) = \int_0^\infty \exp(-t)(P_t q)\,\mathrm{d}t$.

The bilinear symmetric form $(f\,, g) \mapsto \mathbb{E}[\langle Df\,, Dg\rangle_{R_1}]$ is known as a *Dirichlet form*, and the integral $\mathbb{E}[\langle Df\,, Dg\rangle_{R_1}]$ is called the *Dirichlet energy* between f and g. Thus, Theorem 4.7 is another way to state that the covariance between the random variables $f(Z)$ and $g(Z)$ is the Dirichlet energy between the functions f and g.

Let us mention another property of the potential operator.

pr:R1:L **Proposition 4.8** (Hille, XXX and Yoshida, XXX)**.** *The range of R_1 coincides with* Dom$[\mathscr{L}]$, *and*

$$\mathscr{L} f = f - R_1^{-1} f \qquad \textit{a.s. for all } f \in \mathrm{Dom}[\mathscr{L}].$$

Proposition 4.8 is a carefully-crafted way of saying that $\mathscr{L} = I - R_1^{-1}$—equivalently $R_1 = (I - \mathscr{L})^{-1}$—where I denotes the identity operator, $I(f) := f$.

Proof. Choose and fix an arbitrary $f \in R_1(L^2(\mathbb{P}_n))$. There exists $g \in L^2(\mathbb{P}_n)$ such that $f = R_1 g$, equivalently $g = R_1^{-1} f$. Therefore, by (4.11),

$$\mathbb{E}[f\mathscr{H}_k] = \mathbb{E}\left[(R_1 g)\mathscr{H}_k\right] = \frac{\mathbb{E}(g\mathscr{H}_k)}{1+|k|} \quad \text{for all } k \in \mathbb{Z}^n.$$

It follows that

$$R_1^{-1} f = g = \sum_{k\in\mathbb{Z}^n} \frac{\mathbb{E}[g\mathscr{H}_k]}{k!}\,\mathscr{H}_k = \sum_{k\in\mathbb{Z}^n} \frac{1+|k|}{k!}\,\mathbb{E}[f\mathscr{H}_k]\,\mathscr{H}_k. \tag{4.14}$$ R11

Conversely, the preceding infinite sum defines an element of $L^2(\mathbb{P}_n)$ as long as it converges in $L^2(\mathbb{P}_n)$. Consequently,

$$R_1\left(L^2(\mathbb{P}_n)\right) = \left\{ f \in L^2(\mathbb{P}_n) : \sum_{k\in\mathbb{Z}^n} \frac{(1+|k|)^2}{k!}\,\left|\mathbb{E}(f\mathscr{H}_k)\right|^2 < \infty \right\}.$$

For all $k \in \mathbb{Z}^n$, $1 + |k|^2 \leqslant (1+|k|)^2 \leqslant 2(1+|k|^2)$. Therefore,

$$\sum_{k\in\mathbb{Z}^n} \frac{(1+|k|)^2}{k!}\,\left|\mathbb{E}[f\mathscr{H}_k]\right|^2 < \infty \quad \text{iff} \quad \sum_{k\in\mathbb{Z}^n} \frac{|k|^2}{k!}\,\left|\mathbb{E}[f\mathscr{H}_k]\right|^2 < \infty,$$

valid for every $f \in L^2(\mathbb{P}_n)$. This observation and (4.5) together imply that $R_1(L^2(\mathbb{P}_n)) = \mathrm{Dom}[\mathscr{L}]$.

The identity $\mathscr{L} f = f - R_1^{-1} f$ is a consequence of (4.4) and (4.14). □