1. Unordered Selection, continued

Let us recall the following:

**Theorem 7.1.** The number of ways to create a team of \( r \) things among \( n \) is “\( n \) choose \( r \).” Its numerical value is

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}.
\]

**Example 7.2.** If there are \( n \) people in a room, then they can shake hands in \( \binom{n}{2} \) many different ways. Indeed, the number of possible hand shakes is the same as the number of ways we can list all pairs of people, which is clearly \( \binom{n}{2} \). Here is another, equivalent, interpretation. If there are \( n \) vertices in a “graph,” then there are \( \binom{n}{2} \) many different possible “edges” that can be formed between distinct vertices. The reasoning is the same.

**Example 7.3** (Recap). There are \( \binom{52}{5} \) many distinct poker hands.

**Example 7.4** (Poker). The number of different “pairs” \([a, a, b, c, d]\) is

\[
\begin{aligned}
\text{choose the } a & \quad \times \quad \binom{4}{2} \quad \times \quad \binom{12}{3} \quad \times \quad \binom{4}{3} \\
\text{deal the two } a \text{'s} & \quad \text{choose the } b, c, \text{ and } d & \quad \text{deal } b, c, d
\end{aligned}
\]

Therefore,

\[
P(\text{pairs}) = \frac{13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3}{\binom{52}{5}} \approx 0.42.
\]
Example 7.5 (Poker). Let $A$ denote the event that we get two pairs $[a, a, b, b, c]$. Then,

$$|A| = \binom{13}{2} \times \binom{4}{2} \times \binom{13}{2} + \binom{4}{2}.$$ 

Therefore,

$$P(\text{two pairs}) = \frac{\binom{13}{2} \times \binom{4}{2} \times 13 \times 4}{\binom{52}{5}} \approx 0.06.$$ 

Example 7.6. How many subsets does $\{1, \ldots, n\}$ have? Assign to each element of $\{1, \ldots, n\}$ a zero [“not in the subset”] or a one [“in the subset”]. Thus, the number of subsets of a set with $n$ distinct elements is $2^n$. 

Example 7.7. Choose and fix an integer $r \in \{0, \ldots, n\}$. The number of subsets of $\{1, \ldots, n\}$ that have size $r$ is $\binom{n}{r}$. This, and the preceding proves the following amusing combinatorial identity:

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n.$$ 

You may need to also recall the first principle of counting.

The preceding example has a powerful generalization.

Theorem 7.8 (The binomial theorem). For all integers $n \geq 0$ and all real numbers $x$ and $y$,

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}.$$ 

Remark 7.9. When $n = 2$, this yields the familiar algebraic identity

$$(x + y)^2 = x^2 + 2xy + y^2.$$ 

For $n = 3$ we obtain

$$(x + y)^3 = \binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y^1 + \binom{3}{3} x^3 y^0 = y^3 + 3xy^2 + 3x^2 y + x^3.$$
1. Unordered Selection, continued

**Proof.** This is obviously correct for $n = 0, 1, 2$. We use induction. Induction hypothesis: True for $n - 1$.

\[
(x + y)^n = (x + y) \cdot (x + y)^{n-1}
\]

\[
= (x + y) \sum_{j=0}^{n-1} \binom{n-1}{j} x^j y^{n-j-1}
\]

\[
= \sum_{j=0}^{n-1} \binom{n-1}{j} x^j y^{n-j-1} + \sum_{j=0}^{n-1} \binom{n-1}{j} x^j y^{n-j}.
\]

Change variables $[k = j + 1$ for the first sum, and $k = j$ for the second] to deduce that

\[
(x + y)^n = \sum_{k=1}^{n} \binom{n-1}{k-1} x^k y^{n-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}
\]

\[
= \sum_{k=1}^{n-1} \left\{ \binom{n-1}{k-1} + \binom{n-1}{k} \right\} x^k y^{n-k} + x^n + y^n.
\]

But

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}
\]

\[
= \frac{(n-1)!}{(k-1)!(n-k-1)!} \left\{ \frac{1}{n-k} + \frac{1}{k} \right\}
\]

\[
= \frac{(n-1)!}{(k-1)!(n-k-1)!} \times \frac{n}{(n-k)k}
\]

\[
= \frac{n!}{k!(n-k)!} = \binom{n}{k}.
\]

The binomial theorem follows. □