1. Examples of continuous random variables

Example 22.1 (Standard normal density). I claim that

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right)$$

defines a density function. Clearly, $\phi(x) \geq 0$ and is continuous at all points $x$. So it suffices to show that the area under $\phi$ is one. Define

$$A = \int_{-\infty}^{\infty} \phi(x) \, dx.$$

Then,

$$A^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left( -\frac{x^2 + y^2}{2} \right) \, dx \, dy,$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} \exp \left( -\frac{r^2}{2} \right) \, r \, dr \, d\theta.$$

Let $s = r^2/2$ to find that the inner integral is $\int_{0}^{\infty} \exp(-s) \, ds = 1$. Therefore, $A^2 = 1$ and hence $A = 1$, as desired. [Why is $A$ not $-1$?]

The distribution function of $\phi$ is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} \, dz.$$

One can prove that there is “no nice formula” that “describes” $\Phi(x)$ for all $x$ (theorem of Liouville). Usually, people use tables of integrals to evaluate $\Phi(x)$ for concrete values of $x$. 
Example 22.2 (Gamma densities). Choose and fix two numbers (parameters) $\alpha, \lambda > 0$. The gamma density with parameters $\alpha$ and $\lambda$ is the probability density function that is proportional to

$$
\begin{cases}
x^{\alpha-1}e^{-\lambda x} & \text{if } x \geq 0, \\
0 & \text{if } x < 0.
\end{cases}
$$

Now,

$$
\int_0^{\infty} x^{\alpha-1}e^{-\lambda x} \, dx = \frac{1}{\lambda^\alpha} \int_0^{\infty} y^{\alpha-1}e^{-y} \, dy.
$$

Define the gamma function as

$$
\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1}e^{-y} \, dy \quad \text{for all } \alpha > 0.
$$

One can prove that there is “no nice formula” that “describes” $\Gamma(\alpha)$ for all $\alpha$ (theorem of Liouville). Thus, the best we can do is to say that the following is a Gamma density with parameters $\alpha, \lambda > 0$:

$$
f(x) = \begin{cases}
\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x} & \text{if } x \geq 0, \\
0 & \text{if } x < 0.
\end{cases}
$$

You can probably guess by now (and correctly!) that $F(x) = \int_{-\infty}^{x} f(y) \, dy$ cannot be described by nice functions either. Nonetheless, let us finish by making the observation that $\Gamma(\alpha)$ is computable for some reasonable values of $\alpha > 0$. The key to unraveling this remark is the following “reproducing property”:

$$
\Gamma(\alpha + 1) = \alpha \Gamma(\alpha) \quad \text{for all } \alpha > 0. \tag{18}
$$

The proof uses integration by parts:

$$
\Gamma(\alpha + 1) = \int_0^{\infty} x^{\alpha}e^{-x} \, dx
$$

$$
= \int_0^{\infty} u(x)v'(x) \, dx,
$$

where $u(x) = x^\alpha$ and $v'(x) = e^{-x}$. Integration by parts states that\(^1\)

$$
\int uv' = uv - \int v'u \quad \text{for indefinite integrals}.
$$

\(^1\)This follows immediately from integrating the product rule: $(uv)' = u'v + uv'$. 

Evidently, \( u'(x) = \alpha x^{\alpha-1} \) and \( v(x) = -e^{-x} \). Hence,

\[
\Gamma(\alpha + 1) = \int_0^\infty x^\alpha e^{-x} \, dx \\
= u|_0^\infty - \int_0^\infty v'u \\
= (-\alpha x^{\alpha-1} e^{-x})|_0^\infty + \alpha \int_0^\infty x^{\alpha-1} e^{-x} \, dx.
\]

The first term is zero, and the second (the integral) is \( \alpha \Gamma(\alpha) \), as claimed. Now, it easy to see that \( \Gamma(1) = \int_0^\infty e^{-x} \, dx = 1 \). Therefore, \( \Gamma(2) = 1 \times \Gamma(1) = 1 \), \( \Gamma(3) = 2 \times \Gamma(2) = 2 \), \ldots, and in general,

\[
\Gamma(n + 1) = n! \quad \text{for all integers } n \geq 0.
\]

2. Functions of a continuous random variable

The basic problem: If \( Y = g(X) \), then how can we compute \( f_Y \) in terms of \( f_X \)?

**Example 22.3.** Suppose \( X \) is uniform on \((0,1)\), and \( Y = -\ln X \). Then, we compute \( f_Y \) by first computing \( F_Y \), and then using \( f_Y = F_Y' \). Here are the details:

\[
F_Y(a) = P\{Y \leq a\} = P\{-\ln X \leq a\}.
\]

Now, \(-\ln(x)\) is a decreasing function. Therefore, \(-\ln(x) \leq a\) if and only if \( x \geq e^{-a} \), and hence,

\[
F_Y(a) = P\{X \geq e^{-a}\} = 1 - F_X(e^{-a}).
\]

Consequently,

\[
f_Y(a) = -f_X(e^{-a}) \frac{d}{da}(e^{-a}) = e^{-a} f_X(e^{-a}).
\]

Now recall that \( f_X(u) = 1 \) if \( 0 \leq u \leq 1 \) and \( f_X(u) = 0 \) otherwise. Now \( e^{-a} \) is between zero and one if and only if \( a \geq 0 \). Therefore,

\[
f_X(e^{-a}) = \begin{cases} 
1 & \text{if } a \geq 0, \\
0 & \text{if } a < 0.
\end{cases}
\]

It follows then that

\[
f_Y(a) = \begin{cases} 
 e^{-a} & \text{if } a \geq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

Thus, \(-\ln X\) has an exponential density with parameter \( \lambda = 1 \). More generally, if \( \lambda > 0 \) is fixed, then \(-\frac{1}{\lambda}\ln X\) has an exponential density with parameter \( \lambda \).
Example 22.4. Suppose $X$ has density $f_X$. Then let us find the density function of $Y = X^2$. Again, we seek to first compute $F_Y$. Now, for all $a > 0$, 

$$F_Y(a) = P\{X^2 \leq a\} = P\{-\sqrt{a} \leq X \leq \sqrt{a}\} = F_X(\sqrt{a}) - F_X(-\sqrt{a}).$$

Differentiate $[d/da]$ to find that 

$$f_Y(a) = \frac{f_X(\sqrt{a}) + f_X(-\sqrt{a})}{2\sqrt{a}}$$

On the other hand, $f_Y(a) = 0$ if $a \leq 0$. For example, consider the case that $X$ is standard normal. Then, 

$$f_{X^2}(a) = \begin{cases} 
\frac{e^{-a}}{\sqrt{2\pi a}} & \text{if } a > 0, \\
0 & \text{if } a \leq 0.
\end{cases}$$

Or if $X$ is Cauchy, then 

$$f_{X^2}(a) = \begin{cases} 
\frac{1}{\pi \sqrt{a}(1 + a)} & \text{if } a > 0, \\
0 & \text{if } a \leq 0.
\end{cases}$$

Example 22.5. Suppose $\mu \in \mathbb{R}$ and $\sigma > 0$ are fixed constants, and define $Y = \mu + \sigma X$. Find the density of $Y$ in terms of that of $X$. Once again, 

$$F_Y(a) = P\{\mu + \sigma X \leq a\} = P\left\{X \leq \frac{a - \mu}{\sigma}\right\} = F_X\left(\frac{a - \mu}{\sigma}\right).$$

Therefore, 

$$f_Y(a) = \frac{1}{\sigma}f_X\left(\frac{a - \mu}{\sigma}\right).$$

For example, if $X$ is standard normal, then 

$$f_{\mu+\sigma X}(a) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right).$$

This is the so-called $N(\mu, \sigma^2)$ density.

Example 22.6. Suppose $X$ is uniformly distributed on $(0, 1)$, and define 

$$Y = \begin{cases} 
0 & \text{if } 0 \leq X < \frac{1}{3}, \\
1 & \text{if } \frac{1}{3} \leq X < \frac{2}{3}, \\
2 & \text{if } \frac{2}{3} \leq X < 1.
\end{cases}$$

Then, $Y$ is a discrete random variable with mass function, 

$$f_Y(x) = \begin{cases} 
\frac{1}{3} & \text{if } x = 0, 1, \text{ or } 2, \\
0 & \text{otherwise}.
\end{cases}$$
For instance, in order to compute $f_Y(1)$ we note that
\[
f_Y(1) = P\left\{ \frac{1}{3} \leq X < \frac{2}{3} \right\} = \int_{1/3}^{2/3} f_X(y) \, dy = \frac{1}{3}.
\]