1. (Cumulative) distribution functions

Let $X$ be a discrete random variable with mass function $f$. The (cumulative) distribution function $F$ of $X$ is defined by

$$F(x) = P(X \leq x).$$

Here are some of the properties of distribution functions:

1. $F(x) \leq F(y)$ whenever $x \leq y$; therefore, $F$ is non-decreasing.
2. $1 - F(x) = P(X > x)$.
3. $F(b) - F(a) = P(a < X \leq b)$ for $a < b$.
4. $F(x) = \sum_{y : y \leq x} f(y)$.
5. $F(\infty) = 1$ and $F(-\infty) = 0$. [Some care is needed]
6. $F$ is right-continuous. That is, $F(x+) = F(x)$ for all $x$.
7. $f(x) = F(x) - F(x-)$ is the size of the jump [if any] at $x$.

**Example 10.1.** Suppose $X$ has the mass function

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0, \\ \frac{1}{2} & \text{if } x = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Thus, $X$ has equal chances of being zero and one. Define a new random variable $Y = 2X - 1$. Then, the mass function of $Y$ is

$$f_Y(x) = f_X \left( \frac{x + 1}{2} \right) = \begin{cases} \frac{1}{2} & \text{if } x = -1, \\ \frac{1}{2} & \text{if } x = 1, \\ 0 & \text{otherwise}. \end{cases}$$
The procedure of this example actually produces a theorem.

**Theorem 10.2.** If $Y = g(X)$ for a function $g$, then

$$f_Y(x) = \sum_{z : g(z) = x} f_X(z).$$

2. **Expectation**

The *expectation* $EX$ of a random variable $X$ is defined formally as

$$EX = \sum_x xf(x).$$

If $X$ has infinitely-many possible values, then the preceding sum must be defined. This happens, for example, if $\sum_x |xf(x)| < \infty$. Also, $EX$ is always defined [but could be $\pm \infty$] if $P(X \geq 0) = 1$, or if $P(X \leq 0) = 1$. The *mean* of $X$ is another term for $EX$.

**Example 10.3.** If $X$ takes the values $\pm 1$ with respective probabilities $1/2$ each, then $EX = 0$.

**Example 10.4.** If $X = \text{Bin}(n, p)$, then I claim that $EX = np$. Here is why:

$$EX = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k}$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^k q^{(n-1)-(k-1)}$$

$$= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{(n-1)-j}$$

$$= np,$$

thanks to the binomial theorem.
Example 10.5. Suppose \( X = \text{Poiss}(\lambda) \). Then, I claim that \( \mathbb{E}X = \lambda \). Indeed,

\[
\mathbb{E}X = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} \\
= \lambda \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} \\
= \lambda \sum_{j=0}^{\infty} e^{-\lambda} \frac{\lambda^j}{j!} \\
= \lambda,
\]

because \( e^\lambda = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \), thanks to Taylor’s expansion.

Example 10.6. Suppose \( X \) is negative binomial with parameters \( r \) and \( p \). Then, \( \mathbb{E}X = \frac{r}{p} \) because

\[
\mathbb{E}X = \sum_{k=r}^{\infty} k \binom{k-1}{r-1} p^r q^{k-r} \\
= \sum_{k=r}^{\infty} \frac{k!}{(r-1)! (k-r)!} p^r q^{k-r} \\
= r \sum_{k=r}^{\infty} \binom{k}{r} p^r q^{k-r} \\
= \frac{r}{p} \sum_{k=r}^{\infty} \binom{k}{r} p^{r+1} q^{(k+1)-(r+1)} \\
= \frac{r}{p} \sum_{j=r+1}^{\infty} \binom{j-1}{(r+1)-1} p^{r+1} q^{-(r+1)} \\
= \frac{r}{p}.
\]

Thus, for example, \( \mathbb{E}[\text{Geom}(p)] = 1/p \).

Finally, two examples to test the boundary of the theory so far.

Example 10.7 (A random variable with infinite mean). Let \( X \) be a random variable with mass function,

\[
f(x) = \begin{cases} 
\frac{1}{C x^2} & \text{if } x = 1, 2, \ldots, \\
0 & \text{otherwise},
\end{cases}
\]
where $C = \sum_{j=1}^{\infty} (1/j^2)$. Then,

$$EX = \sum_{j=1}^{\infty} j \cdot \frac{1}{C_j^2} = \infty.$$ 

But $P(X < \infty) = \sum_{j=1}^{\infty} 1/(C_j^2) = 1$.

**Example 10.8** (A random variable with an undefined mean). Let $X$ be a random with mass function,

$$f(x) = \begin{cases} 
\frac{1}{Dx^2} & \text{if } x = \pm 1, \pm 2, \ldots, \\
0 & \text{otherwise,} 
\end{cases}$$

where $D = \sum_{j \in \mathbb{Z}\setminus\{0\}} (1/j^2)$. Then, $EX$ is undefined. If it were defined, then it would be

$$\lim_{n,m \to \infty} \left( \sum_{j=-m}^{-1} \frac{j}{Dj^2} + \sum_{j=1}^{n} \frac{j}{Dj^2} \right) = \frac{1}{D} \lim_{n,m \to \infty} \left( \sum_{j=-m}^{-1} \frac{1}{j} + \sum_{j=1}^{n} \frac{1}{j} \right).$$

But the limit does not exist. The rough reason is that if $N$ is large, then $\sum_{j=1}^{N} (1/j)$ is very nearly $\ln N$ plus a constant (Euler’s constant). “Therefore,” if $n, m$ are large, then

$$\left( \sum_{j=-m}^{-1} \frac{1}{j} + \sum_{j=1}^{n} \frac{1}{j} \right) \approx -\ln m + \ln n = \ln \left( \frac{n}{m} \right).$$

If $n = m \to \infty$, then this is zero; if $m \gg n \to \infty$, then this goes to $-\infty$; if $n \gg m \to \infty$, then it goes to $+\infty$. 