Chapter 8 Problems

3. (a) First, we find the value of $c$:

$$c = \frac{1}{\int_0^\infty \int_0^\infty \exp(-x - y) \, dx \, dy} = \frac{1}{\int_0^1 e^{-x} \, dx \cdot \int_0^1 e^{-y} \, dy} = 1.$$ 

(b) Use Figure 1 to find that

$$P\{X + Y > 1\} = \int_0^1 \left( \int_{1-x}^\infty e^{-x-y} \, dy \right) \, dx + \int_1^\infty \left( \int_0^x e^{-x-y} \, dy \right) \, dx$$

$$= \int_0^1 e^{-x} \left( \int_{1-x}^\infty e^{-y} \, dy \right) \, dx + \int_1^\infty e^{-x} \left( \int_0^x e^{-y} \, dy \right) \, dx$$

$$= \int_0^1 e^{-x} \cdot e^{-(1-x)} \, dx + \int_1^\infty e^{-x} \cdot 1 \, dx$$

$$= 2e^{-1} \approx 0.7358.$$

(c) Without integration: $P\{Y > 1 + X\} = 1$, therefore $P\{X < Y\} = 1$.

4. First we calculate $F_Z$: $F_Z(a) = 0$ if $a < 0$ since $X,Y \geq 0$ with probability one (why?). And if $a \geq 0$, then

$$F_Z(a) = P\{X + Y \leq a\}$$

$$= \int_0^a \left( \int_0^a f(x,y) \, dx \right) \, dy \quad \text{(plot the region)}$$

$$= \int_0^a \left( \int_0^a g(x+y) \, dx \right) \, dy.$$
In order to calculate the inner integral: Fix $y$ between zero and $a$ and change variables $[z := x + y]$ to find that

$$\int_0^a g(x + y) \, dx = \int_y^{a+y} g(z) \, dz.$$ 

Therefore [see Figure 2],

$$F_Z(a) = \int_0^a \int_y^{a+y} g(z) \, dz \, dy$$

$$= \int_0^a g(z) \left( \int_0^z dy \right) \, dz + \int_a^{2a} g(z) \left( \int_{z-a}^a dy \right) \, dz$$

$$= \int_0^a zg(z) \, dz + \int_a^{2a} (2a - z)g(z) \, dz$$

$$= \int_0^a zg(z) \, dz + 2a \int_a^{2a} g(z) \, dz - \int_a^{2a} zg(z) \, dz.$$

Differentiate $[d/da]$ to find that if $a < 0$, then $F_Z(a) = 0$; else if $a \geq 0$, then

$$f_Z(a) = ag(a) + 2 \int_a^{2a} g(z) \, dz + 2a \{g(2a) - g(a)\} - 2ag(2a) - ag(a)$$

$$= 2 \int_a^{2a} g(z) \, dz - 2ag(a).$$

[The fundamental theorem of calculus] We can slightly simplify the latter:

$$f_Z(a) = 2 \int_a^{2a} (g(z) - g(a)) \, dz.$$

6. We write

$$w(x, y) = x^2 + y^2 \quad \text{and} \quad z(x, y) = y/x.$$
Then $r := \sqrt{w}$ and $\theta := \arctan(z)$ define $(x, y)$ in polar coordinates. That is,
\[ x = \sqrt{w} \cos(\arctan z) \quad \text{and} \quad y = \sqrt{w} \sin(\arctan z). \]

Then, the Jacobian of this transformation is
\[ J(w, z) = \frac{\partial x}{\partial w} \frac{\partial y}{\partial z} - \frac{\partial x}{\partial z} \frac{\partial y}{\partial w}. \]

But
\[ \frac{\partial x}{\partial w} = \frac{1}{2\sqrt{w}} \cos(\arctan z) \]
\[ \frac{\partial x}{\partial z} = -\sqrt{w} \frac{\sin(\arctan z)}{1 + z^2} \]
\[ \frac{\partial y}{\partial w} = \frac{1}{2\sqrt{w}} \sin(\arctan z) \]
\[ \frac{\partial y}{\partial z} = \sqrt{w} \frac{\cos(\arctan z)}{1 + z^2}. \]

Therefore,
\[ J(w, z) = \frac{1}{2(1 + z^2)}. \]

Consequently,
\[ f_{W,Z}(w, z) = f_{X,Y}(x(w, z), y(w, z)) \times |J(w, z)|. \]

Because $x^2(w, z) + y^2(w, z) = w^2$,
\[ f_{X,Y}(x(w, z), y(w, z)) = \frac{c}{(1 + w^2)^{3/2}}, \]
and so
\[ f_{W,Z}(w, z) = \frac{c}{2(1 + w^2)^{3/2}(1 + z^2)}. \]

for $w \geq 0$ and $-\infty < z < \infty$. Else, $f_{W,Z}(w, z) = 0$. We can write
\[ f_{W,Z}(w, z) = \frac{c\pi}{2(1 + w^2)^{3/2}} \times \frac{1}{\pi(1 + z^2)}, \quad w \geq 0, -\infty < z < \infty. \]

Therefore, $W$ and $Z$ are independent with
\[ f_Z(z) = \frac{1}{\pi(1 + z^2)} \quad \text{(i.e., Z is Cauchy)}, \]
and
\[ f_W(w) = \frac{c\pi}{2(1 + w^2)^{3/2}}, \quad w \geq 0. \]
It remains to find $c$ [this is a part of #5]. Clearly,

$$c = 2 \pi \int_0^\infty \frac{1}{(1 + w^2)^{-3/2}} \, dw.$$

In order to compute the integral, change variables $a = 1/(1 + w^2)$ to find that

$$\int_0^\infty \frac{dw}{(1 + w^2)^{3/2}} = \frac{1}{2} \int_0^1 \frac{da}{\sqrt{1 - a}} = 1.$$

Therefore, $c = 2/\pi$. 
