Chapter 7 Problems

3. Because $\int_0^1 x^\alpha (1-x)^\beta\, dx = B(\alpha+1\,,\beta+1)$ for all $\alpha, \beta \geqslant 0$,

$$EX = \frac{1}{B(a\,,b)} \int_0^1 x^a (1-x)^{b-1}\, dx = \frac{B(a+1\,,b)}{B(a\,,b)}.$$

Similarly,

$$E(X^2) = \frac{1}{B(a\,,b)} \int_0^1 x^{a+1} (1-x)^{b-1}\, dx = \frac{B(a+2\,,b)}{B(a\,,b)}.$$

Therefore,

$$\mathrm{Var}\, X = \frac{B(a+2\,,b)}{B(a\,,b)} - \left(\frac{B(a+1\,,b)}{B(a\,,b)}\right)^2.$$

5. We have

$$f(x) = \frac{1}{B(1/2\,,1/2)} \times \frac{1}{\sqrt{x(1-x)}} \qquad \text{for } 0 < x < 1.$$

Therefore, for all a between zero and one,

$$\begin{aligned} F(a) &= \frac{1}{B(1/2\,,1/2)} \int_0^a \frac{1}{\sqrt{x(1-x)}}\, dx \\ &= \frac{2}{B(1/2\,,1/2)} \int_0^{\sqrt{a}} \frac{1}{\sqrt{1-y^2}}\, dy \qquad\qquad [y := \sqrt{x}] \\ &= \frac{2}{B(1/2\,,1/2)} \arcsin\left(\sqrt{a}\right). \end{aligned}$$

If $a > 1$ then $F(a) = 1$, and if $a < 0$ then $F(a) = 0$. Note that it must be that $F(1) = P\{X \leqslant 1\} = 1$. Therefore, $\arcsin(1) = \frac{1}{2}B(1/2\,,1/2)$. But $\arcsin(1) = \pi/2$; therefore, $B(1/2\,,1/2) = \pi$, and hence

$$f(x) = \begin{cases} \dfrac{1}{\pi\sqrt{x(1-x)}} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F(a) = \begin{cases} 1 & \text{if } a \geqslant 1 \\ \dfrac{2}{\pi} \arcsin\left(\sqrt{a}\right) & \text{if } 0 < a < 1, \\ 0 & \text{otherwise,} \end{cases}$$

7. Recall that
$$F_X(z) = \begin{cases} 1 - e^{-\lambda z} & \text{if } z \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$
Clearly, $Y \geqslant 0$; therefore, $F_Y(z) = 0$ if $z \leqslant 0$. If $z > 0$, then
$$F_Y(z) = P\{Y \leqslant z\} = P\left\{e^{aX} \leqslant z\right\} = P\left\{aX \leqslant \ln z\right\}.$$
I am going to consider the case that $a \geqslant 0$ only; you should think about the case that $a < 0$. In the present setting,
$$F_Y(z) = P\left\{X \leqslant \frac{1}{a}\ln z\right\} = \begin{cases} 1 - e^{-\lambda \ln z/a} & \text{if } \ln z \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$
Note that: (i) $\exp(-\lambda \ln z/a) = z^{-\lambda/a}$; and (ii) $\ln z \geqslant 0$ if and only if $z \geqslant e$. Therefore,
$$F_Y(z) = \begin{cases} 1 - z^{-\lambda/a} & \text{if } z \geqslant e, \\ 0 & \text{otherwise;} \end{cases}$$
and hence
$$f_Y(z) = \begin{cases} \dfrac{\lambda}{a} z^{-1-\lambda/a} & \text{if } z \geqslant e, \\ \\ 0 & \text{otherwise.} \end{cases}$$
Since $EX = (\lambda/a)\int_e^\infty z^{-\lambda/a}\,dz$, $EX < \infty$ if and only if $\lambda > a$.

10. (a) Let $Y := |X|$, and note that $F_Y(a) = 0$ if $a \leqslant 0$. If $a > 0$, then
$$F_Y(a) = P\{-a \leqslant X \leqslant a\} = P\{X \leqslant a\} - P\{X \leqslant -a\} = \Phi(a) - (1 - \Phi(a)) = 2\Phi(a) - 1.$$
(b) The preceding shows that in general, if $Y := |X|$, then $F_Y(a) = 0$ for $a \leqslant 0$; and $F_Y(a) = F_X(a) - F_X(-a)$ for $a > 0$. Thus, when it exists, the density of Y is $f_Y(a) = f_X(a) + f_X(-a)$ for $a > 0$; and zero for $a \leqslant 0$.

23. Let $Y := X^{-2}$ to find that $F_Y(a) = 0$ if $a \leqslant 0$; and
$$F_Y(a) = P\left\{\frac{1}{X^2} \leqslant a\right\} = 1 - P\left\{X^2 \leqslant \frac{1}{a}\right\} \qquad \text{for } a > 0.$$
Thus,
$$F_Y(a) = 2 - 2\Phi\left(\frac{1}{\sqrt{a}}\right) \qquad \text{for } a > 0.$$
(Why?) Differentiate to find that
$$f_Y(a) = \begin{cases} \dfrac{1}{a^{3/2}\sqrt{2\pi}} \exp\left(-\dfrac{1}{2a}\right) & \text{if } a > 0, \\ \\ 0 & \text{otherwise.} \end{cases}$$

35. (a) $\mathsf{E}X = \mathsf{E}\int_0^X \,dx = \mathsf{E}\int_0^\infty I\{X > x\}\,dx = \int_0^\infty P\{X > x\}\,dx$.

(b) $\mathsf{E}(X^r) = \mathsf{E}\int_0^X rx^{r-1}\,dx = \mathsf{E}\int_0^\infty rx^{x-1} I\{X > x\}\,dx = \int_0^\infty rx^{r-1} P\{X > x\}\,dx$.

(c) $e^{\theta X} = 1 + \int_0^X e^{\theta x}\,dx = 1 + \int_0^\infty e^{\theta x} I\{X > x\}\,dx$. Take expectations.