Chapter 5 Problems

2. (a) 0.
   (b) $f(2,2) = \frac{1}{16}$.
   (c) $f(1,2) + f(1,4) + f(2,3) = \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$.
   (d) $P\{X \leqslant 1 + Y\} = 1$, since $P\{X \leqslant Y\} = 1$ by (a).

3. See lecture 16 (Example 16.5).

4. The expectations is $= 0 = \sum_x \sum_y (x-y)^2 f(x,y)$. Therefore, for all pairs (x,y), either $x = y$ or $f(x,y) = 0$ in the double sum. In other words,

$$\sum\sum_{x \neq y} f(x,y) = 0.$$

   But the double sum $= P\{X \neq Y\}$.

5. We know that

$$E\left[(X-Y)^2\right] = E(X^2) + E(Y^2) - 2E(XY).$$

   Therefore, $E(XY) = 0$, which means that X and Y are orthogonal (Definition on page 169). Note that X and Y are uncorrelated if and only if $E(X) \cdot E(Y) = 0$ also. So the "hint" is slightly flawed.

6. (a) If x is an integer in $\{0, \ldots, 4n\}$, then

$$\sin\left(\tfrac{1}{2}\pi x\right) = \begin{cases} 0 & \text{if } x \text{ is even, including } x = 0, \\ 1 & \text{if } x = 1, 5, 9, \ldots, \text{ i.e., } x = 1 + 4k \text{ for some } k \in \{0, \ldots, n-1\}, \\ -1 & \text{if } x = 3, 7, 11, \ldots, \text{ i.e., } x = -1 + 4j \text{ for some } j \in \{1, \ldots, n\}. \end{cases}$$

   Similarly, if x is an integer in $\{0, \ldots, 4n\}$, then

$$\cos\left(\tfrac{1}{2}\pi x\right) = \begin{cases} 0 & \text{if } x \text{ is odd}, \\ 1 & \text{if } x = 4k \text{ for some } k \in \{0, \ldots, n\}, \\ -1 & \text{if } x = -2 + 4j \text{ for some } j \in \{1, \ldots, n\}. \end{cases}$$

   Check by induction that there there are exactly:
   - $n + 1$ values of $x \in \{0, \ldots, 4n\}$ such that $\cos(\frac{1}{2}\pi x) = 1$;
   - $2n$ values of $x \in \{0, \ldots, 4n\}$ such that $\cos(\frac{1}{2}\pi x) = 0$;
   - n values of $x \in \{0, \ldots, 4n\}$ such that $\cos(\frac{1}{2}\pi x) = -1$; and
   - there are $4n + 1$ integers between zero and $4n$.

   When $\cos(\frac{1}{2}\pi x) = \pm 1$, $\sin(\frac{1}{2}\pi x) = 0$ per force. Therefore,

$$f_{Y,Z}(0,1) = P\left\{\cos(\tfrac{1}{2}\pi X) = 1\right\} = \frac{n+1}{4n+1},$$
$$f_{Y,Z}(0,-1) = P\left\{\cos(\tfrac{1}{2}\pi X) = -1\right\} = \frac{n}{4n+1},$$
$$f_{Y,Z}(0,z) = 0 \quad \text{for all other values of } z.$$

   Similarly, there are exactly:

- n values of $x \in \{0, \dots, 4n\}$ such that $\sin(\frac{1}{2}\pi x) = 1$;
- $2n + 1$ values of $x \in \{0, \dots, 4n\}$ such that $\sin(\frac{1}{2}\pi x) = 0$;
- n values of $x \in \{0, \dots, 4n\}$ such that $\sin(\frac{1}{2}\pi x) = -1$; and
- there are $4n + 1$ integers between zero and $4n$.

Therefore,

$$f_{Y,Z}(1, 0) = P\left\{\sin(\tfrac{1}{2}\pi X) = 1\right\} = \frac{n}{4n+1},$$
$$f_{Y,Z}(-1, 0) = P\left\{\sin(\tfrac{1}{2}\pi X) = -1\right\} = \frac{n}{4n+1},$$
$$f_{Y,Z}(y, 0) = 0 \quad \text{for all other values of } y.$$

To summarize, $f_{Y,Z}$ can be tabulated as follows:

$y \backslash z$	-1	0	1
-1	0	$\frac{n}{4n+1}$	0
0	$\frac{n+1}{4n+1}$	0	$\frac{n}{4n+1}$
1	0	$\frac{n}{4n+1}$	0

(b) $X + Y$ is -1 with probability $f_{Y,Z}(-1, 0) + f_{Y,Z}(0, -1) = \frac{n}{4n+1} + \frac{n+1}{4n+1} = \frac{2n+1}{4n+1}$, and $+1$ with probability $\frac{2n}{4n+1}$. Finally, $YZ = 0$, therefore, $E(YZ) = 0$.

7. We have:

$$\begin{aligned}
EU &= aEX + bEY, \\
EV &= cEX + dEY, \qquad \text{and hence} \\
EU \cdot EV &= ac(EX)^2 + (ad + bc)EX \cdot EY + bd(EY)^2, \qquad \text{and also} \\
E(UV) &= E\left[acX^2 + (ad + bc)XY + bdY^2\right] \\
&= acE(X^2) + (ad + bc)E(XY) + bdE(Y^2).
\end{aligned}$$

Therefore,

$$\text{Cov}(U, V) = ac\text{Var}(X) + (ad + bc)\text{Cov}(X, Y) + bd\text{Var}(Y).$$

This shows that U and V are uncorrelated if and only if

$$ac\text{Var}(X) + (ad + bc)\text{Cov}(U, V) + bd\text{Var}(Y) = 0.$$

I.e., either $ad + bc = 0$ and $ac\text{Var}(X) + bd\text{Var}(Y) = 0$, or

$$\text{Cov}(U, V) = -\frac{bd\text{Var}(Y) + ac\text{Var}(X)}{ad + bc}.$$

Therefore, in any event, U and V are uncorrelated if and only if $ac\text{Var}(X) + bd\text{Var}(Y) = 0$. The choice of a, b, c, d is not unique. For instance, we can take $a = c = 1$ and $b = -1$. Then U and V are uncorrelated if and only if $\text{Var}(X) = d\text{Var}(Y)$. Or consider $a = d = 1$ and $c = -1$.

8. The probability that a message gets transmitted correctly is

$$
\begin{aligned}
&\mathrm{P}\{2 \text{ back-to-back wrong transmissions}\} + \mathrm{P}\{2 \text{ back-to-back right transmissions}\} \\
&= (1-p)^2 + p^2 \\
&= 1 - 2p + 2p^2.
\end{aligned}
$$

Call this $h(p)$; since $h'(p) = -2 + 4p$ and $h''(p) = 4 \geqslant 0$, to find that $\min_p h(p) = \frac{1}{2}$, as stated.

For three channels, the probability of correct transmission is

$$
\begin{aligned}
&\mathrm{P}\{0 \text{ incorrect transmissions}\} + \mathrm{P}\{2 \text{ incorrect transmissions}\} \\
&= p^3 + \binom{3}{2}(1-p)^2 p \\
&= p^3 + 3p(1-p)^2 \\
&= 4p^3 - 6p^2 + 3p.
\end{aligned}
$$

This is minimized at $p = 0$, and the thus-minimized probability is zero.