Chapter 4 Problems

3. Let X equal the number of dots. Then, $f(x) = \frac{1}{6}$ if $x = 1, \ldots, 6$; else, $f(x) = 0$. Consequently,

$$\mathrm{E}X = \left(1 \times \frac{1}{6}\right) + \cdots + \left(6 \times \frac{1}{6}\right) = \frac{7}{2}$$
$$\mathrm{E}(X^2) = \left(1^2 \times \frac{1}{6}\right) + \cdots + \left(6^2 \times \frac{1}{6}\right) = \frac{91}{6}.$$

Therefore
$$\operatorname{Var} X = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12} = 2.916\cdots.$$

4. If X is uniform on $\{1, \ldots, n\}$, then $\operatorname{Var} X = (n^2-1)/12$; see "Lecture 12."

11. Because $\mathrm{E}(\{X-a\}^2) = \sum_x (x-a)^2 f(x)$ is assumed to converge nicely, we can differentiate term by term to find that

$$\begin{aligned}\frac{d}{da}\mathrm{E}\left(\{X-a\}^2\right) &= \frac{d}{da}\sum_x (x^2 - 2xa + a^2)f(x)\\ &= \sum_x (-2x+2a)f(x)\\ &= 2a\sum_x f(x) - 2\sum_x xf(x)\\ &= 2a - 2\mathrm{E}X.\end{aligned}$$

Set this equal to zero to find that $a = \mathrm{E}X$. Also,

$$\frac{d^2}{da^2}\mathrm{E}\left(\{X-a\}^2\right) = \sum_x 2f(x) = 2,$$

which is positive. Positive second derivative means the minimum occurs when the derivative is zero; that is, $a = \mathrm{E}X$. But $\mathrm{E}(\{X - \mathrm{E}X\}^2) = \operatorname{Var} X$.

18. By Theorem 4 (p. 121),

$$\begin{aligned}\mathrm{E}\left(\frac{1}{X}\right) &= \sum_{k=1}^{\infty}\frac{1}{k}pq^{k-1} = \frac{p}{q}\sum_{k=1}^{\infty}\frac{1}{k}q^k = \frac{p}{q}\sum_{k=1}^{\infty}\int_0^q x^{k-1}\,dx\\ &= \frac{p}{q}\int_0^q \sum_{k=1}^{\infty} x^{k-1}dx = \frac{p}{q}\int_0^q \frac{1}{1-x}\,dx\\ &= -\frac{p}{q}\ln(1-q) = -\frac{p}{1-p}\ln(p) = \frac{1}{p-1}\ln(p) = \ln\left(p^{\frac{1}{p-1}}\right).\end{aligned}$$

29. (a) "No tail in the first n tosses" means "all heads in the first n tosses." Therefore, probab. = p^n.

(b) $p_n = (1-p)^{n-1}p$.

(c) $E(\text{number}) = \sum_{n=1}^{\infty} np_n = p\sum_{n=1}^{\infty} n(1-p)^{n-1}$, which is equal to

$$-p\frac{d}{dp}\sum_{n=0}^{\infty}(1-p)^n = -p\left(\frac{1}{p}\right)' = \frac{1}{p}.$$