Chapter 4 Problems

3. Let $X$ equal the number of dots. Then, $f(x) = \frac{1}{6}$ if $x = 1, \ldots, 6$; else, $f(x) = 0$. Consequently,

$$EX = \left(1 \times \frac{1}{6}\right) + \cdots + \left(6 \times \frac{1}{6}\right) = \frac{7}{2}$$

$$E(X^2) = \left(1^2 \times \frac{1}{6}\right) + \cdots + \left(6^2 \times \frac{1}{6}\right) = \frac{91}{6}.$$  

Therefore

$$\text{Var } X = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12} = 2.916 \cdots.$$  

4. If $X$ is uniform on $\{1, \ldots, n\}$, then $\text{Var } X = \frac{(n^2 - 1)}{12}$; see “Lecture 12.”

11. Because $E((X - a)^2) = \sum x(x - a)^2f(x)$ is assumed to converge nicely, we can differentiate term by term to find that

$$\frac{d}{da} E((X - a)^2) = \frac{d}{da} \sum_x (x^2 - 2xa + a^2)f(x)$$

$$= \sum_x (-2x + 2a)f(x)$$

$$= 2a \sum_x f(x) - 2 \sum_x xf(x)$$

$$= 2a - 2EX.$$  

Set this equal to zero to find that $a = EX$. Also,

$$\frac{d^2}{da^2} E((X - a)^2) = \sum_x 2f(x) = 2,$$

which is positive. Positive second derivative means the minimum occurs when the derivative is zero; that is, $a = EX$. But $E((X - EX)^2) = \text{Var } X$.

18. By Theorem 4 (p. 121),

$$E\left(\frac{1}{X}\right) = \sum_{k=1}^{\infty} \frac{1}{k} p q^{k-1} = p \sum_{k=1}^{\infty} \frac{1}{k} q^{k} = p \sum_{k=1}^{\infty} \int_0^q x^{k-1} \, dx$$

$$= p \int_0^q \sum_{k=1}^{\infty} x^{k-1} \, dx = p \int_0^q \frac{1}{1-x} \, dx$$

$$= -\frac{p}{q} \ln(1 - q) = -\frac{p}{1-p} \ln(p) = \frac{1}{p-1} \ln(p) = \ln\left(p^{\frac{1}{p-1}}\right).$$
29. (a) “No tail in the first \( n \) tosses” means “all heads in the first \( n \) tosses.” Therefore, probab. = \( p^n \).

(b) \( p_n = (1 - p)^{n-1} p \).

(c) \( E(\text{number}) = \sum_{n=1}^{\infty} np_n = p \sum_{n=1}^{\infty} n(1 - p)^{n-1} \), which is equal to

\[
-p \frac{d}{dp} \sum_{n=0}^{\infty} (1 - p)^n = -p \left( \frac{1}{p} \right)' = \frac{1}{p}.
\]