Chapter 4 Problems

1. (a) Ω = the collection of all possible ways to select three from the 16 grapefruits so described. Note that $|\Omega| = \binom{16}{3} = 560$.

   (b) We need to compute $f(0)$, $f(1)$, $f(2)$, and $f(3)$, since $f(x) = 0$ for all other values of x. Now

$$f(0) = \frac{\binom{4}{3}}{\binom{16}{3}} = \frac{4}{560}.$$

Similarly,

$$f(1) = \frac{\binom{12}{1} \times \binom{4}{2}}{\binom{16}{3}} = \frac{72}{560},$$

$$f(2) = \frac{\binom{12}{2} \times \binom{4}{1}}{\binom{16}{3}} = \frac{264}{560}, \qquad \text{and}$$

$$f(3) = \frac{\binom{12}{3}}{\binom{16}{3}} = \frac{220}{560}.$$

As an aside, we have also

$$E(X) = \left(1 \times \frac{72}{560}\right) + \left(2 \times \frac{264}{560}\right) + \left(3 \times \frac{220}{560}\right) = \frac{1260}{560}.$$

5(a). (i) Want $c \cdot \sum_{x=1}^{\infty} 2^x/x! = 1$, that is to say that c is the reciprocal of $\sum_{x=1}^{\infty} 2^x/x!$. By the Taylor–McLaurin formula $\sum_{x=0}^{\infty} 2^x/x! = e^2$. Therefore, $\sum_{x=1}^{\infty} 2^x/x! = e^2 - 1$, and hence $c = 1/(e^2 - 1)$.

(ii) Want $c = 1/\sum_{x=1}^{\infty} p^x$. Since $\sum_{x=0}^{\infty} p^x = 1/(1-p)$, $\sum_{x=1}^{\infty} p^x = 1/(1-p) - 1 = p/(1-p)$. Therefore, $c = (1-p)/p$.

(iii) As before, $c = 1/\sum_{x=1}^{\infty} p^x x^{-1}$. Now

$$\sum_{x=1}^{\infty} \frac{p^x}{x} = \sum_{x=1}^{\infty} \int_0^p u^{x-1}\,du = \int_0^p \sum_{x=1}^{\infty} u^{x-1}\,du = \int_0^p \frac{du}{1-u} = \ln\left(\frac{1}{1-p}\right).$$

Therefore, c is the reciprocal of $\ln(1/(1-p))$.

(iv) $c = 1/\sum_{x=1}^{\infty} x^{-2}$, which happens to be $6/\pi^2$, since the sum can be shown to be $\pi^2/6$.

(v) c is the reciprocal of $\sum_{x=1}^{\infty} [x(x+1)]^{-1}$. Because

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1},$$

$$\begin{aligned}\sum_{x=1}^{N}\frac{1}{x(x+1)} &= \sum_{x=1}^{N}\frac{1}{x} - \sum_{x=1}^{N}\frac{1}{x+1}\\ &= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}\right) - \left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N+1}\right)\\ &= 1-\frac{1}{N+1}.\end{aligned}$$

Let $N \to \infty$ to find that $\sum_{x=1}^{\infty}[x(x+1)]^{-1} = 1$, and hence $c = 1$.

6. Yes, yes, and yes.

8. Condition on the position of the first tails tossed. Let T_j denote the event that the first tail occurs on the jth toss. Then,

$$\Pr(A_n) = \Pr(A_n \mid T_1)\Pr(T_1) + \Pr(A_n \mid T_2)\Pr(T_2) + \Pr(A_n \mid T_3)\Pr(T_3),$$

since $\Pr(A_n \cap T_4) = \Pr(A_n \cap T_5) = \cdots = 0$. You should check that $\Pr(T_k) = (1/2)^k$. Therefore,

$$\Pr(A_n) = \frac{1}{2}\Pr(A_n \mid T_1) + \frac{1}{4}\Pr(A_n \mid T_2) + \frac{1}{8}\Pr(A_n \mid T_3).$$

Because the tosses are independent from one another, the conditional probab. of A_n, given that the first toss is tails, is the same as $\Pr(A_{n-1})$. Similarly, $\Pr(A_n \mid T_2) = \Pr(A_{n-2})$ and $\Pr(A_n \mid T_3) = \Pr(A_{n-3})$. Therefore,

$$\Pr(A_n) = \frac{1}{2}\Pr(A_{n-1}) + \frac{1}{4}\Pr(A_{n-2}) + \frac{1}{8}\Pr(A_{n-3}). \qquad (*)$$

Because $\Pr(A_1) = \Pr(A_2) = 0$ and $\Pr(A_3) = 1/8$, the preceding inductive formula solves $\Pr(A_n)$ for all $n \geqslant 4$, viz.,

(a) $\Pr(A_4) = \frac{1}{2}\Pr(A_3) = 1/16$;
(b) $\Pr(A_5) = \frac{1}{2}\Pr(A_4) + \frac{1}{4}\Pr(A_3) = 1/16$;
(c) $\Pr(A_6) = \frac{1}{2}\Pr(A_5) + \frac{1}{4}\Pr(A_4) + \frac{1}{8}\Pr(A_3) = 1/16$,
(d) $\Pr(A_7) = \frac{1}{2}\Pr(A_6) + \frac{1}{4}\Pr(A_5) + \frac{1}{8}\Pr(A_4) = 7/128$,

etc. [It is true that ($*$) can be solved in terms of the root of a cubic equation, using the generating-function methods of chapter 3, but ($*$) is a very useful formula.]