Chapter 2 Problems

1. Let $W = \{\text{windy}\}$ and $H = \{\text{hit}\}$. We know:

   - $\Pr(H \mid W) = 0.4$;
   - $\Pr(H \mid W^c) = 0.7$;
   - $\Pr(W) = 0.3$.

   (a) $\Pr(W \cap H) = \Pr(H \mid W)\Pr(W) = 0.4 \times 0.3 = 0.12$.

   (b) Write $\Pr(H) = \Pr(H \mid W)\Pr(W) + \Pr(H \mid W^c)\Pr(W^c)$ to find that

   $$\Pr(H) = (0.4 \times 0.3) + (0.7 \times 0.7) = 0.61.$$

   (c) Let $H_j = \{\text{hit on } j\text{th shot}\}$, were $j = 1, 2$. We want

   $$\Pr(H_1 \cap H_2^c) + \Pr(H_1^c \cap H_2).$$

   If we assume that the behavior of the wind, and the hitting of the shots are all independent, from shot to shot, then the preceding is equal to $\Pr(H_1)\Pr(H_2^c) + \Pr(H_1^c)\Pr(H_2) = (0.61 \times 0.39) + (0.39 \times 0.61) = 0.4758$.

   (d) We want $\Pr(W^c \mid H^c) = 1 - \Pr(W \mid H^c)$. First recall that $\Pr(H^c) = 1 - 0.61 = 0.39$. Therefore,

   $$\Pr(W \mid H^c) = \Pr(H^c \mid W) \times \frac{\Pr(W)}{\Pr(H^c)} = 0.6 \times \frac{0.3}{0.39} \approx 0.4615385.$$

2. (a) By definition, $\Pr(A \mid B) = \Pr(A \cap B)/\Pr(B)$. If B is a subset of A, then $A \cap B = B$, and hence $\Pr(A \mid B) = \Pr(B)/\Pr(B) = 1$.

   (b) Similar to (a), but now $A \cap B = A$.

4. (a) Since $E \cap E = E$, it follows that $\Pr(E) = \{\Pr(E)\}^2$. The equation $x = x^2$ has only two solutions: One of them is $x = 0$; if $x \neq 0$, then we can divide that equation by x to find that $x = 1$.

   (b) If $\Pr(A), \Pr(B) > 0$, then $\Pr(A \mid B) = \Pr(A \cap B)/\Pr(B) = 0 \neq \Pr(A)$. Therefore, A and B are *not* independent. If one of them, say A, has zero probability, then $\Pr(A \cap B) = \Pr(A)\Pr(B)$, so they are independent [but this is uninteresting, since A is an "impossible" event in any event].

   (c) If A and B are independent, then $\Pr(A^c \mid B) = 1 - \Pr(A \mid B) = 1 - \Pr(A) = \Pr(A^c)$, and hence A^c and B are also independent. An application of this very fact itself shows that A^c and B^c are also independent [change labels].

10. Let G_j denote the even that the jth child is a girl. We observe that $A \cap B$ denotes the event that this family has exactly one girl and three boys. Therefore,

$$\begin{aligned}
\Pr(A \cap B) &= \Pr(G_1 \cap G_2^c \cap G_3^c \cap G_4^c) + \Pr(G_1^c \cap G_2 \cap G_3^c \cap G_4^c) \\
&\quad + \Pr(G_1^c \cap G_2^c \cap G_3 \cap G_4^c) + \Pr(G_1^c \cap G_2^c \cap G_3^c \cap G_4) \\
&= p(1-p)^3 + (1-p)p(1-p)^2 + (1-p)^2 p(1-p) + (1-p)^3 p \\
&= 4p(1-p)^3.
\end{aligned}$$

11. Let F_j denote the event that these parents have j children; we know that $\Pr(F_1) = \Pr(F_2) = \Pr(F_3) = 1/3$. If $G = \{\text{at least one girl}\}$, then

$$\begin{aligned}
\Pr(G \mid F_1) &= \frac{1}{2} \\
\Pr(G \mid F_2) &= 1 - \Pr(G^c \mid F_2) = \frac{3}{4} \\
\Pr(G \mid F_3) &= 1 - \Pr(G^c \mid F_3) = \frac{7}{8}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Pr(G) &= \Pr(G \mid F_1)\Pr(F_1) + \Pr(G \mid F_2)\Pr(F_2) + \Pr(G \mid F_3)\Pr(F_3) \\
&= \left(\frac{1}{2} \times \frac{1}{3}\right) + \left(\frac{3}{4} \times \frac{1}{3}\right) + \left(\frac{7}{8} \times \frac{1}{3}\right) = \frac{1}{3}\left(\frac{1}{2} + \frac{3}{4} + \frac{7}{8}\right) \\
&= \frac{17}{24}.
\end{aligned}$$

We use the preceding to find

$$\begin{aligned}
\Pr(F_1 \mid G) &= \Pr(G \mid F_1)\frac{\Pr(F_1)}{\Pr(G)} = \frac{1}{2} \times \frac{1/3}{17/24} = \frac{4}{17} \\
\Pr(F_2 \mid G) &= \Pr(G \mid F_2)\frac{\Pr(F_2)}{\Pr(G)} = \frac{3}{4}\frac{1/3}{17/24} = \frac{6}{17} \\
\Pr(F_3 \mid G) &= \Pr(G \mid F_3)\frac{\Pr(F_3)}{\Pr(G)} = \frac{7}{8}\frac{1/3}{17/24} = \frac{7}{17}.
\end{aligned}$$

If $N = \{\text{no older girl}\}$, then

$$\Pr(N \mid F_1 \cap G) = 1. \tag{1}$$

In order to compute $\Pr(N \mid F_2 \cap G)$ let $b_1 g_2$ define the event that the first child is a boy, the second is a girl, and so on. Then,

$$\begin{aligned}
\Pr(N \mid F_2 \cap G) &= \Pr(N \cap b_1 g_2 \mid F_2 \cap G) + \Pr(N \cap g_1 b_2 \mid F_2 \cap G) \\
&\quad + \Pr(N \cap g_1 g_2 \mid F_2 \cap G).
\end{aligned}$$

Since $b_1 g_2 \cap G = b_1 g_2$,

$$\begin{aligned}\Pr(N \cap b_1 g_2 \mid F_2 \cap G) &= \Pr(N \mid b_1 g_2 \cap F_2 \cap G) \times \Pr(b_1 g_2 \mid F_2 \cap G) \\ &= 1 \times \frac{\Pr(b_1 g_2 \mid F_2)}{\Pr(G \mid F_2)} = \frac{1/4}{1 - \frac{1}{4}} = \frac{1}{3}.\end{aligned}$$

Similarly,

$$\Pr(N \cap g_1 b_2 \mid F_2 \cap G) = \frac{1}{3},$$

and

$$\Pr(N \cap g_1 g_2 \mid F_2 \cap G) = \frac{1}{6}.$$

Therefore,

$$\Pr(N \mid F_2 \cap G) = \frac{1}{3} + \frac{1}{3} + \frac{1}{6} = \frac{5}{6}. \tag{2}$$

In order to compute $\Pr(N \mid F_3 \cap G)$ we appeal to symmetry to simplify things a little at first, viz.,

$$\begin{aligned}\Pr(N \mid F_3 \cap G) &= 3 \Pr(N \cap b_1 b_2 g_3 \mid F_3 \cap G) + 3 \Pr(N \cap b_1 g_2 g_3 \mid F_3 \cap G) \\ &\qquad + \Pr(N \cap g_1 g_2 g_3 \mid F_3 \cap G).\end{aligned} \tag{3}$$

Now:

$$\begin{aligned}\Pr(N \cap b_1 b_2 g_3 \mid F_3 \cap G) &= \Pr(N \mid b_1 b_2 g_3 \cap F_3 \cap G) \times \Pr(b_1 b_2 g_3 \mid F_3 \cap G) \\ &= 1 \times \frac{\Pr(b_1 b_2 b_3 \mid F_3)}{\Pr(G \mid F_3)} = \frac{1/8}{7/8} = \frac{1}{7};\end{aligned}$$

$$\begin{aligned}\Pr(N \cap b_1 g_2 g_3 \mid F_3 \cap G) &= \Pr(N \mid b_1 g_2 g_3 \cap F_3 \cap G) \times \Pr(b_1 g_2 g_3 \mid F_3 \cap G) \\ &= \frac{1}{2} \times \frac{\Pr(b_1 g_2 g_3 \mid F_3)}{\Pr(G \mid F_3)} \\ &= \frac{1}{2} \times \frac{1/8}{7/8} = \frac{1}{14}; \quad \text{and}\end{aligned}$$

$$\begin{aligned}\Pr(N \cap g_1 g_2 g_3 \mid F_3 \cap G) &= \Pr(N \mid g_1 g_2 g_3 \cap F_3 \cap G) \times \Pr(g_1 g_2 g_3 \mid F_3 \cap G) \\ &= \frac{1}{3} \times \frac{\Pr(g_1 g_2 g_3 \mid F_3)}{\Pr(G \mid F_3)} \\ &= \frac{1}{3} \times \frac{1/8}{7/8} = \frac{1}{21}.\end{aligned}$$

Therefore, (3) tells us that

$$\Pr(N \mid F_3 \cap G) = \frac{3}{7} + \frac{3}{14} + \frac{1}{21} = \frac{29}{42}.$$

This, (1), and (2) together imply the following:

$$\begin{aligned}\Pr(N \mid G) &= \sum_{j=1}^{3} \Pr(N \cap F_j \mid G) = \sum_{j=1}^{3} \Pr(N \mid F_j \cap G) \Pr(F_j \mid G) \\ &= \left(1 \times \frac{4}{17}\right) + \left(\frac{5}{6} \times \frac{6}{17}\right) + \left(\frac{29}{42} \times \frac{7}{17}\right).\end{aligned}$$

20. Let $S = \{\text{score} = 2\}$ and $T = \{\text{first toss was tails}\}$. Then, $\Pr(S \mid T) = 1/6$. If we toss a coin five times, independently, then the chances are $5 \times (1/2)^5$ that we toss exactly one head and four tails. Therefore, $\Pr(S \mid T^c) = 5 \times (1/2)^5$, and hence

$$\Pr(S) = \Pr(S \mid T) \Pr(T) + \Pr(S \mid T^c) \Pr(T^c) = \frac{31}{192}.$$

Consequently,

$$\Pr(T \mid S) = \Pr(S \mid T) \frac{\Pr(T)}{\Pr(S)} = \frac{1}{6} \times \frac{1/2}{31/192} = \frac{16}{31}.$$

31. Let D denote the event that the document is in box number 1; also define S denote the event that a search of box 1 discovers the document. We know that $\Pr(S \mid D) = p_1$.

(a) Since

$$\Pr(S) = \Pr(S \mid D) \Pr(D) + \Pr(S \mid D^c) \Pr(D^c) = \left(p_1 \times \frac{1}{3}\right) + 0 = \frac{p_1}{3},$$

it follows that

$$\Pr(D \mid S^c) = \Pr(S^c \mid D) \frac{\Pr(D)}{\Pr(S^c)} = (1 - p_1) \times \frac{1/3}{1 - \frac{p_1}{3}} = \frac{1 - p_1}{3 - p_1}.$$

(b) Let F denote the event that box one is searched twice and the document has not been found. By independence, $\Pr(F \mid D) = (1 - p_1)^2$, $\Pr(F \mid D^c) = 1$, and

$$\begin{aligned}\Pr(F) &= \Pr(F \mid D) \Pr(D) + \Pr(F \mid D^c) \Pr(D^c) \\ &= \left((1 - p_1)^2 \times \frac{1}{3}\right) + \frac{2}{3} = \frac{3 - 2p_1 + p_1^2}{3}.\end{aligned}$$

Therefore,

$$\Pr(D \mid F) = \Pr(F \mid D) \frac{\Pr(D)}{\Pr(F)} = \frac{(1 - p_1)^2}{3 - 2p_1 + p_2^2}.$$

(c) Let T denote the event that all three boxes were searched one and the document was not found. Let D_j denote the event that the document is in box j [D and D_1 denote the same event]. Then,

$$\begin{aligned}
\Pr(\mathsf{T}) &= \Pr(\mathsf{T} \mid D_1)\Pr(D_1) + \Pr(\mathsf{T} \mid D_2)\Pr(D_2) + \Pr(\mathsf{T} \mid D_3)\Pr(D_3) \\
&= \frac{1-p_1}{3} + \frac{1-p_2}{3} + \frac{1-p_3}{3} = 1 - \bar{p},
\end{aligned}$$

where $\bar{p} = (p_1 + p_2 + p_3)/3$ is the average the p_j's. Therefore,

$$\Pr(D \mid \mathsf{T}) = \Pr(\mathsf{T} \mid D)\frac{\Pr(D)}{\Pr(\mathsf{T})} = (1 - p_1)\frac{1/3}{1-\bar{p}} = \frac{1-p_1}{3-3\bar{p}}.$$