Solutions to Midterm #4
Mathematics 5010–1, Spring 2006
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1. *A random vector $(X\,,Y)$ has the following (joint) mass function $p(x\,,y) = P\{X = x\,,\; Y = y\}$:*

$y \backslash x$	**1**	**2**	**3**
1	1/3	10/27	1/27
2	1/9	1/27	p
3	1/27	1/27	1/27

(a) *Compute p.*

Solution: The table entries have to add up to one. Therefore,

$$\begin{aligned} 1 &= \frac{1}{3} + \frac{10}{27} + \frac{1}{27} + \frac{1}{9} + \frac{1}{27} + p + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} \\ &= 1 + p. \end{aligned}$$

Therefore, $p = 0$.

(b) *Compute EX.*

Solution: The easiest thing to do is to first find p_X [the marginal mass function of X]. This is given by "adding over the y's:

$y \backslash x$	**1**	**2**	**3**
1	1/3	10/27	1/27
2	1/9	1/27	0
3	1/27	1/27	1/27
$\boldsymbol{p_X(x)}$	13/27	12/27	2/27

Consequently,

$$EX = \left(1 \times \frac{13}{27}\right) + \left(2 \times \frac{12}{27}\right) + \left(3 \times \frac{2}{27}\right) = \frac{43}{27} \approx 1.593.$$

2. *A random vector $(X\,,Y)$ has (joint) density function*

$$f(x\,,y) = \begin{cases} x+y, & \textit{if } 0<x<1 \textit{ and } 0<y<1,\\ 0, & \textit{otherwise.}\end{cases}$$

(a) *Compute the conditional density of X given that $Y = 1/2$.*

Solution: First, we note that for all y between zero and one,

$$f_Y(y) = \int_0^1 (x+y)\,dx = \frac{1}{2} + y.$$

If y is not between 0 and 1, then $f_Y(y) = 0$. Consequently,

$$f_{X|Y}\left(x \mid \tfrac{1}{2}\right) = \begin{cases} x+\frac{1}{2}, & \text{if } 0<x<1,\\ 0, & \text{otherwise,}\end{cases}$$

(b) *Compute $E[X \mid Y = 2]$.*

Solution: This is not defined: You cannot condition on events that are not well defined.

3. (Bonus Problem) *Let $X_1, X_2, \ldots$ be independent random variables, all having the same density function f.*

(a) *Compute $P\{X_1 < X_2\}$.*

Solution: Note that $P\{X_1 < X_2\} = P\{X_1 > X_2\}$. [Write both probabilities out as two-dimensional integrals, for instance.] Because $P\{X_1 = X_2\} = 0$, this implies readily that $P\{X_1 < X_2\} = 1/2$.

(b) *Compute $P\{X_1 < X_2 < \ldots < X_n\}$ for an arbitrary integer $n \geq 2$.*

Solution: Once again, we can note that for any permutation $i_1 < \cdots < i_n$ of $1, \ldots, n$,

$$P\left\{X_{i_1} < \cdots < X_{i_n}\right\} = P\left\{X_1 < \cdots < X_n\right\}.$$

Because there are $n!$ distinct such permutations, $P\{X_1 < \cdots < X_n\} = 1/n!$.

4. *Let X and Y be independent, standard normal random variables. Recall that their common density is the function*

$$f(a) = \frac{1}{\sqrt{2\pi}} e^{-a^2/2}, \qquad -\infty < a < \infty.$$

(a) *Compute the (joint) density function of the random vector $(X\,,Y)$.*

Solution: By independence,

$$f(x\,,y) = f_X(x) f_Y(y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}.$$

(b) *Compute the density of the random variable Z, where*

$$Z := \sqrt{X^2 + Y^2}.$$

Solution: The fastest solution that I can think of is the following:
First we compute F_Z. If $a \le 0$ then $F_Z(a) = 0$. Else, if $a > 0$, then

$$\begin{aligned}
F_Z(a) &= P\left\{\sqrt{X^2+Y^2} \le a\right\} = P\left\{X^2+Y^2 \le a^2\right\} \\
&= \frac{1}{2\pi} \iint\limits_{x^2+y^2 \le a^2} e^{-(x^2+y^2)/2}\, dx\, dy \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^a e^{-r^2/2} r\, dr \right) d\theta \qquad \text{(polar coordinates)} \\
&= 1 - e^{-a^2/2}.
\end{aligned}$$

Then, differentiate to find that

$$f_Z(a) = \begin{cases} ae^{-a^2/2}, & \text{if } a \ge 0, \\ 0, & \text{otherwise.} \end{cases} \qquad \text{(eq.1)}$$

Another, more straight-forward method, is this: First use the convolution formula to find $F_{X^2+Y^2}$. Next, notice that $F_Z(a) = F_{X^2+Y^2}(a^2)$. Finally, differentiate to obtain f_Z. We have computed $F_{X^2+Y^2}$ in the lectures. Here is a reminder: $F_{Y^2}(b)$ if $b \le 0$ and if $b > 0$,

$$F_{Y^2}(b) = P\left\{-\sqrt{b} \le Y \le \sqrt{b}\right\} = \Phi(\sqrt{b}) - \Phi(-\sqrt{b}) = 2\Phi(\sqrt{b}) - 1.$$

Therefore,

$$f_{Y^2}(b) = \begin{cases} 2\Phi'\left(\sqrt{b}\right) \times \frac{1}{2\sqrt{b}}, & \text{if } b > 0, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} \frac{1}{\sqrt{2\pi b}} e^{-b/2}, & \text{if } b > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The function f_{X^2} is the same. Therefore, $f_{X^2+Y^2}(s) = 0$ if $s \le 0$ and else if $s > 0$ then

$$\begin{aligned}
f_{X^2+Y^2}(s) &= \int_{-\infty}^{\infty} f_{X^2}(b) f_{Y^2}(s-b)\, db = \frac{1}{2\pi} \int_0^s \frac{e^{-b/2}}{\sqrt{b}} \times \frac{e^{-(s-b)/2}}{\sqrt{(s-b)}}\, db \\
&= \frac{1}{2} e^{-s/2}.
\end{aligned}$$

Therefore, if $b \ge 0$ then

$$F_{X^2+Y^2}(b) = \frac{1}{2} \int_0^b e^{-s/2}\, ds = 1 - e^{-b/2}.$$

This leads to

$$F_Z(b) = F_{X^2+Y^2}(b^2) = 1 - e^{-b^2/2}.$$

We saw this already in the first part of the solution to this problem. Differentiate to obtain (eq.1) and finish.