Solutions to Midterm #3
Mathematics 5010–1, Spring 2006
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1. *A continuous random variable X has density function f given by the following:*

$$f(x) = \begin{cases} Ce^{-x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) *Compute C.*
Solution: $1 = C\int_0^\infty e^{-x}\,dx = C$, so $C = 1$.

(b) *Find $P\{X > 10\}$.*
Solution: $P\{X > 10\} = \int_{10}^\infty e^{-x}\,dx = e^{-10} \approx 0.000045$.

2. *A salesman has scheduled two appointments to see encyclopedias. His first appointment leads to a sale with probability* 0.3*, and his second with probability* 0.6 *independently of the outcome of the first appointment. Any sale made is equally likely to be either for the deluxe model which costs \$1,000, or the standard model which costs \$500. Let X denote the total value of all the salesman's sales. Compute EX.*

Solution: For $i = 1, 2$ consider the events $S_i := \{\text{sale on the } i\text{th appointment}\}$. We know that S_1 and S_2 are independent, $P(S_1) = 0.3$, and $P(S_2) = 0.6$. Let $D_i :=$ "deluxe on ith", also. We know that $P(D_i \mid S_i) = P(D_i^c \mid S_i) = 1/2$. Consequently, $P(S_i \cap D_i) = P(S_i)/2$ and $P(S_i \cap D_i^c) = P(S_i)/2$.

The possible values of X are:

- 2000 dollars. In this case, we have $P\{X = 2000\} = P(S_1 \cap D_1)P(S_2 \cap D_2) = \frac{0.3}{2} \times \frac{0.6}{2} = 0.045$;
- 1500 dollars. In this case, we have

$$\begin{aligned} P\{X = 1500\} &= P(S_1 \cap D_1)P(S_2 \cap D_2^c) + P(S_1 \cap D_1^c)P(S_2 \cap D_2) \\ &= \left(\frac{0.3}{2} \times \frac{0.6}{2}\right) + \left(\frac{0.3}{2} \times \frac{0.6}{2}\right) = 0.09. \end{aligned}$$

- 1000 dollars. In this case, we have

$$\begin{aligned} P\{X = 1000\} &= P(S_1 \cap D_1)P(S_2^c) + P(S_1^c)P(S_2 \cap D_2) + P(S_1 \cap D_1^c)P(S_2 \cap D_2^c) \\ &= \left(\frac{0.3}{2} \times 0.4\right) + \left(0.7 \times \frac{0.6}{2}\right) + \left(\frac{0.3}{2} + \frac{0.6}{2}\right) = 0.315. \end{aligned}$$

- 500 dollars. In this case, we have

$$\begin{aligned} P\{X = 500\} &= P(S_1 \cap D_1^c)P(S_2^c) + P(S_1^c)P(S_2 \cap D_2^c) \\ &= \left(\frac{0.3}{2} \times 0.4\right) + \left(0.7 \times \frac{0.6}{2}\right) = 0.27. \end{aligned}$$

- 0 dollars. In this case, we have $P\{X = 500\} = P(S_1^c)P(S_2^c) = 0.7 \times 0.4 = 0.28$.

Therefore,

$$EX = (2000 \times 0.045) + (1500 \times 0.09) + (1000 \times 0.315) + (500 \times 0.27) = 675 \text{ dollars.}$$

3. *Suppose X is a uniform $(0\,, 1)$ random variable. Then compute $E[X^n]$ for any integer $n \geq 1$.*
Solution: Evidently, $E[X^n] = \int_0^1 x^n\,dx = 1/(n+1)$.

4. *Suppose X is normally distributed with $\mu = 1$ and $\sigma^2 = 4$. Then compute $P\{X \geq 0\}$.*
Solution: Let Φ denote the standard-normal distribution function. Then, by standardization,

$$P\{X < 0\} = \Phi\left(\frac{0-1}{2}\right) = \Phi(-0.5) = 1 - \Phi(0.5).$$

Therefore, $P\{X \geq 0\} = \Phi(0.5) \approx 0.6915$, thanks to the normal table.

5. *Let X be a random variable with density function*

$$f(x) = \begin{cases} \dfrac{3}{x^4}, & \textit{if } x > 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Compute the mean and variance of X.
Solution: We have

$$\begin{aligned} EX &= \int_1^\infty \frac{3}{x^3}\,dx = \frac{3}{2}, \\ E[X^2] &= \int_1^\infty \frac{3}{x^2}\,dx = 3, \\ \mathrm{Var}X &= E[X^2] - (EX)^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}. \end{aligned}$$