# Math 5010–1, Fall 2006
# Solutions to the Final Examination

1. *Compute the mass function of the random variable $Y$ whose moment generating function is*

$$M_Y(t) = \frac{1}{2}e^t + \frac{1}{6}e^{-2t} + \frac{1}{3}e^{5t}.$$

**Solution:** $p(1) = 1/2$, $p(-2) = 1/6$, $p(5) = 1/3$, and $p(x) = 0$ otherwise.

2. *Consider a random vector $(X\,,Y)$. We know that $X$ is exponentially distributed with parameter $1$; i.e.,*

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & \textit{otherwise.} \end{cases}$$

*Let $x > 0$ be any fixed positive number, and suppose that conditionally on the event $\{X = x\}$, $Y$ is exponentially distributed with parameter $1/x$. That is,*

$$f_{Y|X}(y\,|\,x) = \begin{cases} \dfrac{1}{x}e^{-y/x}, & y \geq 0, \\ 0, & \textit{otherwise}, \end{cases}$$

*Find the density function of $(X^2\,,Y^2)$.*

**Solution:** The joint density of $(X\,,Y)$ is

$$f_{X,Y}(x\,,y) = f_X(x)f_{Y|X}(y|x) = \begin{cases} \dfrac{e^{-x-(y/x)}}{x}, & \text{if } x, y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = g_1(X\,,Y) = X^2$ and $V = g_2(X\,,Y) = Y^2$. The Jacobian of the transformation $g$ is $4xy$. Therefore,

$$f_{U,V}(u\,,v) = \frac{f_{X,Y}(x\,,y)}{4xy} = \begin{cases} \dfrac{e^{-\sqrt{u}-(\sqrt{v}/\sqrt{u})}}{4u\sqrt{v}}, & \text{if } u, v \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

3. *You have 4 light bulbs whose lifetimes are independent normal random variables with mean 100 (hours) and standard deviation 5 (hours). Find the probability that your 4 light bulbs together live for at least 402 hours.*

**Solution:** Let $X_i$ denote the lifetime of the $i$th light bulb. Then $X_i$ is $N(100\,,25)$ for each $i$. Therefore, by independence, $X_1 + \cdots + X_4$ is $N(400\,,100)$. Thus,

$$P\{X_1 + \cdots + X_4 \geq 402\} = P\left\{N(0\,,1) \geq \frac{402-400}{\sqrt{100}}\right\} \approx P\{N(0\,,1) \geq 0.2\} \approx 0.4207.$$

4. *Suppose $X_1, X_2, X_3$ and $X_4$ are independent with common mean $1$ and common variance $2$. Compute $Cov(X_1 + X_2\,, X_2 + X_3)$.*

**Solution:** The covariance is equal to

$$\text{Cov}(X_1\,,X_2) + \text{Cov}(X_1\,,X_3) + \text{Cov}(X_2\,,X_2) + \text{Cov}(X_2\,,X_3) = 0 + 0 + 2 + 0 = 2.$$

5. *Let $X_1, X_2, \ldots, X_{20}$ be independent Poisson random variables with mean one. Use the central limit theorem to approximate $P\{\sum_{i=1}^{20} X_i > 15\}$.*

**Solution:** The expectation of each $X_i$ is 1, and so is the variance. Therefore, $E(\sum_{i=1}^{20} X_i) = 20$, and so is the variance. If we apply the CLT, then

$$P\left\{\sum_{i=1}^{20} X_i > 15\right\} \approx P\left\{N(20\,,20) > 15\right\} = P\left\{N(0\,,1) > \frac{15-20}{\sqrt{20}}\right\} \approx P\left\{N(0\,,1) > -1.1\right\}.$$

This is $1 - \Phi(-1.1) = \Phi(1.1) \approx 0.8643$.