Bayes' Formula (Thomas Bayes, ~~17~~00's) $\overset{-1800}{}$

$$P(E) = \sum_{j=1}^{N} P(E|F_j)\,P(F_j) \qquad F_j \text{ disjoint}; \quad \bigcup_{j=1}^{N} F_j = S.$$

e.g., $P(E) = P(E|F)P(F) + P(E|F^c)\,P(F^c)$.

~~Theo~~

Theorem (Bayes' Formula) If F_j disj't and $\bigcup_{j=1}^{N} F_j = S$, then $\forall E \subseteq S$,

$$P(F_j|E) = \frac{P(E|F_j)\,P(F_j)}{\sum_{k=1}^{N} P(E|F_k)\,P(F_k)}$$

Pf - $\sum_{k=1}^{N} P(E|F_k)P(F_k) = P(E)$, so this follows from the switching lemma: $P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$. #

Example (3k, p.79) A new couple, known to have 2 children, has just moved into town. We've seen the mother with a girl. Find prob. both girls?

$G_1 = \{1^{st} \text{ is girl}\}$; $G_2 = \{2^{nd} \text{ is girl}\}$
$G = \{\text{the child seen is girl}\}$.

B's with "boys". Question $P(G_1 \cap G_2 | G) = ?$

$$P(G_1G_2|G) = \frac{P(G_1 \cap G_2 \cap G)}{P(G)} = \frac{P(G_1 \cap G_2)}{P(G)}$$

$$= \frac{P(G_1 \cap G_2)}{\underbrace{P(G|G_1G_2)}_{=1} P(G_1G_2) + P(G|G_1B_2)P(G_1B_2) + \cdots + \underbrace{P(G|B_1B_2)}_{=0}P(B_1B_2)}$$

Suppose all 4 gender possibilities are equally likely.

$$\Rightarrow \quad P(G_1G_2|G) = \frac{1}{1 + P(G|G_1B_2) + P(G|B_1G_2)}.$$

Need these; could be anything.

① E.g., Irrespective of the gender, the mother takes the older child with prob $p \in [0,1]$. Then,

$$P[G|G_1B_2] = p$$
$$P[G|B_1G_2] = 1-p.$$

Then, $$P[G_1G_2|G] = \frac{1}{1+p+1-p} = \frac{1}{2}.$$

② If mother walks the girl with some prob. q, then

$$P[G|G_1B_2] = q = P[G|B_1G_2], \text{ so}$$

$$P[G_1G_2|G] = \frac{1}{1+2q}.$$

The problem depends on the "dependence" structure between the sex of the child walked and the age of the child walked.

(Statistical) Independence.

A and B are <u>independent</u> if $\mathbb{P}(A|B) = \mathbb{P}(A)$.

$$\Leftrightarrow \quad \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

$$\Leftrightarrow \quad \mathbb{P}(B|A) = \mathbb{P}(B).$$

<u>Ex.</u> $E = \begin{cases} 0 \\ 1 \\ 2 \end{cases}$ $\quad F = \begin{cases} 3\pi \\ 10 \\ 16 \end{cases}$ [E and F are expt's w/ these possible values]

"Joint" prob.'s

E \ F	3π	10	16
0	1/3	0	0
1	0	1/3	0
2	0	0	1/3

← prob's

E and F are <u>dependent</u>.

$$\mathbb{P}(E=0 \cap F=10) = 0$$

$$\mathbb{P}(E=0) = \mathbb{P}(E=0 \cap F=3\pi) = 1/3$$
$$\mathbb{P}(F=10) = \mathbb{P}(E=2 \cap F=10) = 1/3$$

$$0 = \mathbb{P}(E=0 \cap F=10) \neq \mathbb{P}(E=0)\,\mathbb{P}(F=10) = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}.$$

<u>Def</u>: three events E, F, and G are independent if E,F ; F,G ; and E,G are. Extends to $E_1, \ldots, E_N$ in the same way.

Remark Can be that E, F ; F, G ; and E, G are independent but E, F, G is not !

Ex. Toss 2 coins independently (see below).

$E = \{1^{st} \text{ coin} : H\}$

$F = \{2^{nd} \text{ coin} : H\}$

$G = \{\text{Both coins toss the same}\}$.

$$P(E) = 1/2 \qquad P(F) = 1/2$$

$$P(G) = P(\underset{1\,2}{HH}) + P(\underset{1\,2}{TT}) = \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2} = \frac{1}{2}.$$

$$P(G|E) = P(H_2) = \frac{1}{2} = P(G) \quad \Rightarrow \quad G \perp\!\!\!\perp E$$

$$P(G|F) = P(H_1) = \frac{1}{2} \Rightarrow G \perp\!\!\!\perp F$$

$$P(F|E) = P(F) \quad \text{of course.}$$

But $$P(EFG) = P(EF) = \frac{1}{4} \neq P(E)P(F)P(G) = \frac{1}{8}\,!$$

<u>Ex</u>. 2 tosses of a coin
All outcomes equally likely

$$\mathbb{P}(H_1 H_2) = \frac{1}{4} = \mathbb{P}(H_1)\,\mathbb{P}(H_2)$$

$$\mathbb{P}(T_1 T_2) = \frac{1}{4} = \mathbb{P}(T_1)\,\mathbb{P}(T_2)$$

$$\mathbb{P}(T_1 H_2) = \frac{1}{4} = \mathbb{P}(T_1)\,\mathbb{P}(H_2)$$

$$\mathbb{P}(H_1 T_2) = \frac{1}{4} = \mathbb{P}(H_1)\,\mathbb{P}(T_2).$$

$\Rightarrow$ Exp'ts are <u>independent</u>! (+ <u>Vice</u> <u>versa</u>.)

<u>EX</u>. n tosses of a coin, independent tosses

$$\mathbb{P}(\text{run of } n \text{ T's}) = \frac{1}{2^n} \qquad \underline{\text{or}}$$

$$= \left(\frac{1}{2}\right)^n = \frac{1}{2^n}.$$

<u>EX</u>. Toss a p-coin (i.e., $\mathbb{P}(H) = p$) until a H.

Find $\mathbb{P}(n \text{ tosses were required}) = \mathbb{P}(T_1 \cdots T_{n-1} H_n)$

$$= (1-p)^{n-1}\,p.$$

<u>EX</u>. roll a die until a '6'. $\mathbb{P}(n \text{ rolls needed}) = \left(\frac{5}{6}\right)^{n-1} \frac{1}{6}$.
(p-coin $p = 1/6$)

<u>EX</u>. Bin. trials ; n indep't trials ; $\mathbb{P}(\text{success}) = p$ in each trial.

$$P[\text{exactly } k \text{ successes}] = \binom{n}{k} p^k (1-p)^{n-k}.$$