Two Physical Descriptions of Poisson pmf's

1) The law of rare events.

Theorem If $X_n : Bin(n, \frac{\lambda}{n})$ for a fixed $\lambda > 0$, then for all $k = 0, 1, \ldots$,

$$\lim_{n \to \infty} P(X_n = k) = \frac{e^{-\lambda} \lambda^k}{k!}.$$

Application: Proportion of a certain property (eg. lottery, disease, ...) is $\frac{1.5}{10{,}000}$. In a random sample of size 10,000,

$$P[\text{2 people have this property}] \simeq \frac{e^{-1.5}\, 1.5^2}{2!}$$

$$= 0.251021.$$

The exact prob is $\binom{10000}{2} \left(\frac{1.5}{10000}\right)^2 \left(1 - \frac{1.5}{10000}\right)^{9998}$

$$= 0.2510434.$$

Lemma. As $n \to \infty$, $\left(1 + \frac{x}{n}\right)^n \longrightarrow e^x$, $\forall x \in \mathbb{R}$.

Pf $\ln\left(1 + \frac{x}{n}\right)^n = n \ln\left(1 + \frac{x}{n}\right)$.

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \cdots$$

$$\ln\left(1 + \frac{x}{n}\right)^n = n\left[\frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} \cdots\right] = x - \frac{x^2}{2n} + \frac{x^3}{3n^2} \cdots$$

Want to say $\frac{1}{n}[\cdots] \longrightarrow 0$. Why and how? [Taylor w/ remainder.] #

Pf of Th^m $\forall k \geq 0$, $\forall n \geq k$

$$P\{X_n = k\} = \frac{n!}{(n-k)!\ k!} \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k}.$$

a) $$\left(1-\frac{\lambda}{n}\right)^{n-k} = \left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k} \longrightarrow e^{-\lambda}$$

b) $$\frac{n!}{(n-k)!\ n^k} = \frac{n(n-1)\cdots(n-k+1)}{n^k}$$

$$= \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-k+1}{n}\right)$$

$$\longrightarrow 1. \quad \#$$

Colorful examples: (some taken from p.150)

- The # of misprints on page (or a few pages) of a book
- The # of people, in our community, who live 100^+ years
- The # of wrong telephone #'s dialed in a day
- The # of fever-related deaths in a day in the U.S.
- The # of lottery-winners in a given township.

2) Rare-arrivals processes (Somewhat informal)

"Events" occur, as time goes by, that are "rare" in the following sense:

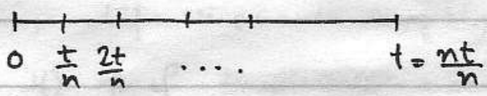
i) In a time-interval of lgth h, the prob of getting exactly 1 occurrence is $\sim \lambda h$ for a fixed $\lambda > 0$.

ii) The prob of getting 2 or more there is $\ll h$.

iii) The # of occurances in all disjoint time-intervals are indep't.

$N(t) :=$ # of events that occur by time t. Then,

$$P\{N(t) = k\} = ?$$



Then $P\{N(t) = k\} = P$[k of these n intervals have exactly 1 event-occurances and the other $n-k$ do not have any]

$+ P$[$N(t) = k$ and at least 1 of these subint's has 2 or more occurances]

$$:= T_1 + T_2$$

$$T_2 \le P[\text{ at least 1 has 2 or more}]$$

$$\le P[1^{st} \text{ has 2 or more}] + P[2^{nd} \text{ has 2 or more}] + \dots + P(n^{th} \text{ has 2 or more})$$

$\cancel{\le n\, o(\tfrac{t}{n})}$

$$<< n \frac{t}{n} = t$$

i.e., $T_2 \longrightarrow 0$ as $n \longrightarrow \infty$.

"Also," $T_1 \cong P[\ Bin(n, \frac{\lambda t}{n}) = k] \xrightarrow{n \to \infty} \frac{e^{-\lambda} \lambda^k}{k!}$, by law of rare events.

$P[N(t) = k]$ does not depend on n. So

$$P[N(t) = k] = \frac{e^{-\lambda} \lambda^k}{k!} \quad !$$

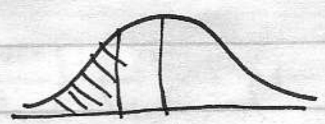
Colorful Examples (some from p 156)

1) # of earthquakes during a given period
2) # of electrons emitted from a heated cathode for a given period.
3) # of deaths, in a given period, of the policy holders of a given life-insurance company
4) # of e-messages on a given e-bulletin board, in a given period.

One more approximation theorem for Binomials (de Moivre, Laplace)

Thm If p is fixed, and $X_n \sim \mathrm{Bin}(n,p)$, then

$$P\left\{ \frac{X_n - np}{\sqrt{np(1-p)}} \le x \right\} \xrightarrow{n \to \infty} \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du .$$



Proof is too hard for now. Much more is done later on.

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