Expectations $\quad E(X) = \int_{-\infty}^{\infty} x f(x)\,dx. \quad \left(\int_{-\infty}^{\infty} |x| f(x)\,dx < \infty\right)$

Examples

① $X \sim \text{Unif}(0,1)$; $f(x) = 1$, $0 \le x \le 1$

$E(X) = \int_0^1 x\,dx = 1/2$.

② $X \sim \text{Exp}(\lambda)$; $f(x) = \lambda e^{-\lambda x}$, $x \ge 0$.

$E(X) = \int_0^\infty x \cdot \lambda e^{-\lambda x}\,dx = 1/\lambda \int_0^\infty y e^{-y}\,dy$

Integration by Parts : $(uv)' = u'v + v'u$

$\Rightarrow \int u'v = uv - \int uv'$.

$$\left.\begin{matrix} u = y & v' = e^{-y} \\ u' = 1 & v = -e^{-y} \end{matrix}\right\} \quad E(X) = 1/\lambda \left[-y e^{-y} \Big|_0^\infty + \int_0^\infty e^{-y}\,dy \right] = 1/\lambda .$$

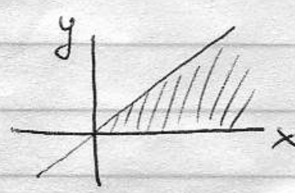
③ $X \sim N(0,1)$; $f(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$; $E(X) = 0$ (symmetry).

Prop $\quad E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx \quad$ if $\int_{-\infty}^{\infty} |g(x)| f(x)\,dx < \infty$.

Suppose $g \ge 0$.

"**pf**" $\quad E[g(X)] = \int_0^\infty y\, f_{g(X)}(y)\,dy \qquad$ (cheat alert!)

$$= \int_0^\infty \left(\int_0^y dx \right) f_{g(X)}(y)\,dy$$



$$= \int_0^\infty \left(\int_x^\infty f_{g(X)}(y)\,dy \right) dx$$

$$= \int_0^\infty P\{g(X) > x\}\,dx .$$

$$= \int_0^\infty \int_{g \ge x} f_X(a)\,da \quad dx = \int_0^\infty \left(\int_0^{g(a)} dx \right) f_X(a)\,da$$

$$\cancel{= E[g(X)]} = \int_0^\infty g(a) f_X(a)\,da . \qquad \#$$

Cor $E(\alpha X + \beta) = \alpha E(X) + \beta$.

Pf $g(x) = \alpha x + \beta$. #

As before, $\mathrm{var}(X) \triangleq E[(X - EX)^2] = E(X^2) - [E(X)]^2$.

Examples ① $X \sim \mathrm{Unif}(0,1)$; $EX^2 = \int_0^1 x^2 dx = 1/3$

$\Rightarrow \mathrm{var}X = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

② $X \sim \mathrm{Exp}(\lambda)$; $EX^2 = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \frac{1}{\lambda^2} \int_0^\infty y^2 e^{-y} dy$.

$$\left.\begin{array}{ll} u = y^2 & v' = e^{-y} \\ u' = 2y & v = -e^{-y} \end{array}\right\} \Rightarrow EX^2 = \frac{1}{\lambda^2}\left[-y^2 e^{-y}\Big|_0^\infty + 2\int_0^\infty y e^{-y} dy \right]$$

$$= 2/\lambda^2$$

$\Rightarrow \mathrm{var}X = 1/\lambda^2$.

③ $X \sim N(0,1)$; $E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx$

$$\left.\begin{array}{ll} u = x & u' = 1 \\ v' = x e^{-x^2/2} & v = -e^{-x^2/2} \end{array}\right\} \Rightarrow$$

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \left\{ -x e^{-x^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2} dx \right\} = 1$$

$\Rightarrow \mathrm{var}(X) = 1$.

Other Normals

$X \sim N(0,1)$

$\mu \in \mathbb{R}, \ \sigma > 0$

$Y = \sigma X + \mu.$

$$F_Y(a) = P\{\sigma X + \mu \le a\} = P\left\{X \le \frac{a-\mu}{\sigma}\right\} = F_X\left(\frac{a-\mu}{\sigma}\right)$$

$$\Rightarrow \quad f_Y(a) = \frac{1}{\sigma} f_X\left(\frac{a-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(a-\mu)^2/2\sigma^2}$$

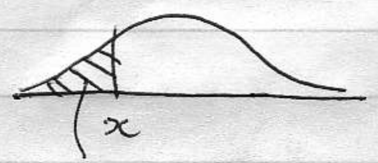
$N(\mu, \sigma^2)$ - pdf. $\qquad E(Y) = \sigma E(X) + \mu = \mu.$

$$\operatorname{var} Y = \operatorname{Var}(\sigma X + \mu) = E\left[(\sigma X + \mu - \mu)^2\right] = \sigma^2 E X^2 = \sigma^2.$$

Using Normal Tables (Table 5.1, p. 203)

if $Y \sim N(\mu, \sigma^2)$ then $X = \frac{Y-\mu}{\sigma} \sim N(0,1)$.

$F_X(x) \equiv \Phi(x)$ is tabulated:



$\Phi(x)$

E.g.,

$\Phi(0) = 0.5$

$\Phi(0.01) = 0.5040$

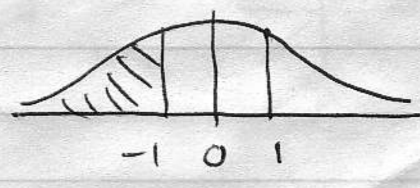
$\Phi(0.05) = 0.5199$

$\Phi(1.0) = 0.8413$

$\Phi(2.0) = 0.9772$

$\Phi(2.68) = 0.9963$

$\vdots$



Also, $\Phi(-1) = 1 - \Phi(1)$

$= 1 - 0.8413$

$= 0.1587$

etc.