Expectations

"weighted average" $\qquad E\underline{X} = \sum_a a\, p(a) = \sum_a a\, P\{\underline{X} = a\}$.

Ex $\quad \underline{X}$ = # of H's in 3 indept tosses of a fair coin.

$$\left.\begin{array}{l} p(0) = p(3) = 1/8 \\ p(1) = p(2) = 3/8 \end{array}\right\} \Rightarrow \quad E\underline{X} = 1.5$$

Ex $\quad \underline{X}$ is $Bin(n,p)$

$$E\underline{X} = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{[(n-1)-(k-1)]!\,(k-1)!} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

$$= np \underbrace{\sum_{j=0}^{n-1} P\{Bin(n-1, p) = j\}}_{1}$$

$$= np$$

Ex. $\underline{X} = Poisson(\lambda) \Rightarrow$

$$E\underline{X} = \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda.$$

Ex. X = a neg. binomial r.v. (r, p); ie.,

$$p(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r} \qquad r = k, k+1, \cdots$$

$$E(X) = \sum_{k=r}^{\infty} k \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

$$= \sum_{k=r}^{\infty} \frac{k!}{(r-1)!\,(k-r)!} p^r (1-p)^{k-r}$$

$$= \frac{r}{p} \cdot \sum_{k=r}^{\infty} \frac{k!}{r!\,(k-r)!} p^{r+1} (1-p)^{(k+1)-(r+1)}$$

$$= \frac{r}{p} \sum_{j=r+1}^{\infty} \binom{j-1}{(r+1)-1} p^{r+1} (1-p)^{j-(r+1)}$$

$$= r/p.$$

Ex (Indicator Function) $1_A = \begin{cases} 1, & \text{if } A \\ 0, & \text{if } A^c \end{cases}$.

$$P(1_A = 1) = P(A) \longrightarrow E(1_A) = P(A).$$

Functions of a r.v.

$Y = X^2$ is a r.v. $\quad P_Y(k) = ?$, $EY = ?$, etc.

$\underline{\mathcal{E}x}$. $X = \begin{cases} 1 \\ -1 \end{cases}$ wp $\frac{1}{2}$ each $(EX = 0)$

$Y = X^2 \equiv 1$ wp 1. $(EY = 1)$

$\underline{\mathcal{E}x}$. $X = 0, \pm 1, \pm 2$ wp $\frac{1}{5}$ each. $Y = X^2$

$$Y = \begin{cases} 0 & 1/5 \\ 1 & 2/5 \\ 4 & 2/5 \end{cases} \Rightarrow EY = EX^2 = \frac{10}{5} = 2.$$

So $E(X^2) \neq (EX)^2$; in fact there is no obvious relation between $E[g(X)]$ and $E(X)$ ... __ever__!

To compute $E[g(X)]$, the 1st method is to find the prob. dist. of $Y = g(X)$, and proceed. This is too arduous. Alternatively, use

__Theorem__ If X has possible values $x_1, x_2, \ldots$ with prob's $p(x_1), p(x_2), \ldots$ then

$$E[g(X)] = \sum_{i=1}^{\infty} g(x_i)\, p(x_i).$$

__Proof__ $E[g(X)] = \sum_{i \geq 1} y_i\, P\{g(X) = y_i\}$ $\qquad \{y_i\}$ = different values of $g(x_j)$

$$= \sum_{i \geq 1} y_i \sum_{j:\, g(x_j) = y_i} P\{X = x_j\} = \sum_j \sum_{i:\, y_i = g(x_j)} y_i\, p(x_j)$$

$$= \sum_j \sum_{i:\, y_i = g(x_j)} g(x_j)\, p(x_j)$$

$$= \sum_j g(x_j)\, p(x_j). \quad \#$$

Cor If $a, b \in \mathbb{R}$ then $E[aX+b] = aE(X) + b$.

Proof $$E(aX+b) = \sum_{i\geq 1} (ax_i + b)\, p(x_i)$$

$$= a \sum_{i \geq 1} x_i\, p(x_i) + b \sum_{i\geq 1} p(x_i)$$

$$= a\,EX + b. \quad *$$

Ex $X : \mathrm{Bin}(n, p)$

$$EX^2 = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k^2 \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \frac{n}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} (k-1) \frac{n!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$+ \sum_{k=1}^{n} \frac{n!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$\cancel{= n \cdot p^2 \sum_{k=2}^{n} \frac{n-2}{(n-k)!\,(k-2)!} p^{k-2} (1-p)^{(n-2)-(k-2)}}$$

$$T_1 = \sum_{k=2}^{n} \frac{(n-2)!}{((n-2)-(k-2))!\,(k-2)!} p^{k-2} (1-p)^{(n-2)-(k-2)} \cdot (n-1)\,n\,p^2$$

$$= \sum_{j=0}^{n-2} \binom{n-2}{j} p^j (1-p)^{n-2-j}\; n(n-1)p^2 = np^2(n-1)$$

$$T_2 = \sum_{k=1}^{n} \frac{n!}{(n-(k-1))!\,(k-1)!}\, p^{k-1}(1-p)^{(n-1)-(k-1)} \cdot p$$

$$= np \sum_{k=1}^{n} \binom{n-1}{(n-1)-(k-1)} p^{k-1}(1-p)^{(n-1)-(k-1)}$$

$$= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^{j}(1-p)^{n-1-j} = np.$$

$$\therefore \quad EX^2 = n(n-1)p^2 + np = n^2p^2 - np^2 + np$$

$$= n^2p^2 + np[1-p] = (EX)^2 + np(1-p).$$