Solutionjs to Problem Assignment #9
Math 501–1, Spring 2006
University of Utah

Problems:

1. *A player throws a fair die and simultaneously flips a fair coin. If the coin lands heads then she wins twice, and if tails, then one-half of the value that appears on the die. Determine her expected winnings.*

Solution: Let $H = \{\text{heads}\}$, and define N to be the number of dots on the rolled die. We know that N and H are independent. Let W denote the amount won. We know that

$$P(W = 2N \mid H) = 1 \quad \text{and} \quad P(W = N/2 \mid H^c) = 1.$$

Therefore,

$$\begin{aligned} E(W) &= E(W \mid H)P(H) + E(W \mid H^c)P(H^c) \\ &= E(2N \mid H)P(H) + E\left(\frac{N}{2} \,\middle|\, H^c\right) P(H^c) \\ &= E(2N)P(H) + E\left(\frac{N}{2}\right) P(H^c), \end{aligned}$$

by independence. But $P(H) = P(H^c) = 1/2$, and $E(N) = (1 + \cdots + 6)/6 = 7/2$. Consequently,

$$E(W) = 2E(N)P(H) + \frac{1}{2}E(N)P(H^c) = \frac{35}{8} = 4.375.$$

2. *If X and Y are independent uniform-$(0\,,1)$ random variables, then prove that*

$$E\left(|X - Y|^\alpha\right) = \frac{2}{(\alpha+1)(\alpha+2)} \qquad \text{for all } \alpha > 0.$$

Solution: The density of $(X\,,Y)$ is $f(x\,,y) = 1$ if $0 \leq x, y \leq 1$; $f(x\,,y) = 0$, otherwise. Therefore,

$$\begin{aligned} E\left(|X - Y|^\alpha\right) &= \int_0^1 \int_0^1 |x - y|^\alpha \, dx \, dy \\ &= \int_0^1 \int_x^1 (y - x)^\alpha \, dy \, dx + \int_0^1 \int_0^x (x - y)^\alpha \, dy \, dx \\ &= \int_0^1 \int_0^{1-x} z^\alpha \, dz \, dx + \int_0^1 \int_0^x w^\alpha \, dw \, dx \\ &= \frac{1}{1+\alpha} \int_0^1 (1 - x)^{\alpha+1} \, dx + \frac{1}{1+\alpha} \int_0^1 x^{\alpha+1} \, dx \\ &= \frac{2}{(1+\alpha)(2+\alpha)}, \end{aligned}$$

as asserted.

3. *A group of n men and n women are lined up at random.*

(a) *Find the expected number of men who have a woman next to them.*

Solution: All $(2n)!$ possible permutations are equally likely. Now consider the events $M_i = \{\text{person } i \text{ is a man}\}$, for $i = 1, \ldots, 2n$. Clearly, $P(M_i) = 1/2$ [produce the requisite combinatorial argument]. Let W_i denote the number of women who are neighboring person i. We can note that if $i = 2, \ldots, n-1$, then

$$P(W_i = 0 \,|\, M_i) = \frac{P(W_i = 0\ ,\ M_i)}{P(M_i)} = \frac{P(M_{i-1} \cap M_i \cap M_{i+1})}{1/2}$$
$$= 2P(M_{i-1} \cap M_i \cap M_{i+1}) = 2\frac{\binom{n}{3}(2n-3)!}{(2n)!}.$$

Consequently, for $i = 2, \ldots, n-1$,

$$P(W_i \geq 1\,, M_i) = P(W_i \geq 1 \,|\, M_i)P(M_i) = \left(1 - 2\frac{\binom{n}{3}(2n-3)!}{(2n)!}\right) \times \frac{1}{2} := \mathcal{A}.$$

Similarly,

$$P(W_1 \geq 1\,, M_1) = P(W_n \geq 1\,, M_n) = \frac{n^2(2n-2)!}{(2n)!} := \mathcal{B}.$$

Let $I_i = 1$ if M_i occurs and $W_i \geq 1$. Evidently, the expected number of men who have a woman next to them is $\sum_{i=1}^n I_i$. Note that $E(I_i) = P(W_i \geq 1\,, M_i)$. Therefore,

$$E\left(\sum_{i=1}^{2n} I_i\right) = \sum_{i=1}^{2n} E(I_i)$$
$$= P(W_1 \geq 1\,, M_1) + \sum_{i=2}^{2n-1} P(W_i \geq 1\,, M_i) + P(W_n \geq 1\,, M_n)$$
$$= 2\mathcal{B} + (2n-2)\mathcal{A}.$$

(b) *Repeat part* **(a)***, but now assume that the group is randomly seated at a round table.*

Solution: The difference now is that $P(W_i \geq 1\,, M_i) = \mathcal{A}$ for all $i = 1, \ldots, n$, so that

$$E\left(\sum_{i=1}^{2n} I_i\right) = 2n\mathcal{A}.$$

4. *Let $X_1, X_2, \ldots$ be independent with common mean μ and common variance σ^2. Set*

$$Y_n = X_n + X_{n+1} + X_{n+2} \qquad \textit{for all } n \geq 1.$$

Compute $\mathrm{Cov}(Y_n\,, Y_{n+j})$ *for all* $n \geq 1$ *and* $j \geq 0$.

Solution: Note that

$$\begin{aligned} E(Y_n) &= E(X_n) + E(X_{n+1}) + E(X_{n+2}) = 3\mu, \\ E(Y_{n+j}) &= 3\mu, \quad \text{thus,} \\ E(Y_n)\cdot E(Y_{n+j}) &= 9\mu^2. \end{aligned}$$

Also,

$$\begin{aligned} Y_n &= X_n + X_{n+1} + X_{n+2}, \\ Y_{n+j} &= X_{n+j} + X_{n+j+1} + X_{n+j+2}. \end{aligned}$$

Thus,

$$\begin{aligned} Y_nY_{n+j} = &\ X_nX_{n+j} + X_nX_{n+j+1} + X_nX_{n+j+2} \\ &+ X_{n+1}X_{n+j} + X_{n+1}X_{n+j+1} + X_{n+1}X_{n+j+2} \\ &+ X_{n+2}X_{n+j} + X_{n+2}X_{n+j+1} + X_{n+2}X_{n+j+2}. \end{aligned}$$

Take expectations to find that

$$\begin{aligned} E\left(Y_nY_{n+j}\right) = &\ E\left(X_nX_{n+j}\right) + E\left(X_nX_{n+j+1}\right) + E\left(X_nX_{n+j+2}\right) \\ &+ E\left(X_{n+1}X_{n+j}\right) + E\left(X_{n+1}X_{n+j+1}\right) + E\left(X_{n+1}X_{n+j+2}\right) \\ &+ E\left(X_{n+2}X_{n+j}\right) + E\left(X_{n+2}X_{n+j+1}\right) + E\left(X_{n+2}X_{n+j+2}\right). \end{aligned}$$

Let us work this out in separate cases, depending on the value of $j \geq 0$. First, consider the case that $j = 0$. Then,

$$\begin{aligned} E\left(Y_n^2\right) = &\ E\left(X_n^2\right) + E\left(X_nX_{n+1}\right) + E\left(X_nX_{n+2}\right) \\ &+ E\left(X_{n+1}X_n\right) + E\left(X_{n+1}^2\right) + E\left(X_{n+1}X_{n+2}\right) \\ &+ E\left(X_{n+2}X_n\right) + E\left(X_{n+2}X_{n+1}\right) + E\left(X_{n+2}^2\right). \end{aligned}$$

But $E(X_n^2) = E(X_{n+1}^2) = E(X_{n+2}^2) = \mathrm{Var}(X_n) + (EX_n)^2 = \sigma^2 + \mu^2$. Also, if $n \neq m$, then by independence $E(X_nX_m) = E(X_n)E(X_m) = \mu^2$. Therefore,

$$E\left(Y_n^2\right) = 3\left(\sigma^2 + \mu^2\right) + 6\mu^2 = 3\sigma^2 + 9\mu^2. \qquad (j = 0)$$

Next, consider the case that $j = 1$. In this case,

$$\begin{aligned} E\left(Y_nY_{n+1}\right) &= E\left(X_nX_{n+1}\right) + E\left(X_nX_{n+2}\right) + E\left(X_nX_{n+3}\right) \\ &\quad + E\left(X_{n+1}^2\right) + E\left(X_{n+1}X_{n+2}\right) + E\left(X_{n+1}X_{n+3}\right) \\ &\quad + E\left(X_{n+2}X_{n+1}\right) + E\left(X_{n+2}^2\right) + E\left(X_{n+2}X_{n+3}\right) \\ &= 7\mu^2 + 2\left(\sigma^2 + \mu^2\right) \\ &= 2\sigma^2 + 9\mu^2. \qquad (j = 1) \end{aligned}$$

Next we consider the case $j = 2$. In this case,

$$\begin{aligned} E\left(Y_nY_{n+2}\right) &= E\left(X_nX_{n+2}\right) + E\left(X_nX_{n+3}\right) + E\left(X_nX_{n+4}\right) \\ &\quad + E\left(X_{n+1}X_{n+2}\right) + E\left(X_{n+1}X_{n+3}\right) + E\left(X_{n+1}X_{n+4}\right) \\ &\quad + E\left(X_{n+2}^2\right) + E\left(X_{n+2}X_{n+3}\right) + E\left(X_{n+2}X_{n+4}\right) \\ &= \sigma^2 + 9\mu^2. \qquad (j=2) \end{aligned}$$

Finally, if $j \geq 3$, then

$$E\left(Y_nY_{n+j}\right) = 9\mu^2. \qquad (j \geq 3)$$

Theoretical Problems:

1. *Suppose X is a nonnegative random variable with density function f. Prove that*

$$E(X) = \int_0^\infty P\{X > t\}\, dt. \qquad \text{(eq.1)}$$

Is this still true when $P\{X < 0\} > 0$? If "yes," then prove it. If "no," then construct an example.

Solution: The trick is to start with the right-hand side:

$$\begin{aligned} \int_0^\infty P\{X > t\}\, dt &= \int_0^\infty \int_t^\infty f(x)\, dx\, dt = \int_0^\infty \int_0^x f(x)\, dt\, dx \\ &= \int_0^\infty x f(x)\, dx = E(X). \end{aligned}$$

This cannot be true when $P\{X < 0\} > 0$. For instance, suppose $f(x) = \frac{1}{2}$ if $-1 \leq x \leq 1$, and $f(x) = 0$ otherwise. [f is the uniform-$(-1\,, 1)$ density.] Then, $E(X) = 0$, whereas the preceding shows that $\int_0^\infty P\{X > t\}\, dt = \int_0^\infty x f(x)\, dx = (1/2) \int_0^1 x\, dx = (1/4)$.

2. (Hard) *Suppose $X_1, \ldots, X_n$ are independent, and have the same distribution. Then, compute $\phi(x)$ for all x, where*

$$\phi(x) := E\left[X_1 \mid X_1 + \cdots + X_n = x\right].$$

Solution: Note that the distribution of $(X_1\,, \ldots, X_n)$ is the same as $(X_2\,, X_1, \ldots, X_n)$. Therefore, $\phi(x) = E[X_2 \,|\, X_1 + \cdots + X_n = x]$ as well. Similarly, $\phi(x) = E[X_3 \,|\, X_1 + \cdots + X_n = x] = \cdots = E[X_n \,|\, X_1 + \cdots + X_n = x]$. Add the preceding equations to find that

$$n\phi(x) = E\left[X_1 + \cdots + X_n \mid X_1 + \cdots + X_n = x\right] = x.$$

Thus, $\phi(x) = (x/n)$.