Solutions to Assignment #3
Math 501–1, Spring 2006
University of Utah

Problems:

1. *What is the probability that at least one of a pair of fair dice lands on 6, given that the sum of the dice is i, for $i = 2, 3, \ldots, 12$?*

Solution: Let A denote the event that at least one of a pair of fair dice lands on 6, and S_i the event that the sum of the dice is i, for $i = 2, 3, \ldots, 12$. It suffices to do this for $i = 7, 8, 9, 10, 11, 12$, because the others are zero. First, we compute $P(S_i)$ for the latter values of i. But $P(S_i)$ is $\#(S_i)/36$, where $\#(S_i)$ denotes the number of ordered ways of getting the sum to be i. Thus,

$$\begin{aligned}
P(S_7) &= 6/36 \\
P(S_8) &= 5/36 \\
P(S_9) &= 4/36 \\
P(S_{10}) &= 3/36 \\
P(S_{11}) &= 2/36 \\
P(S_{12}) &= 1/36.
\end{aligned}$$

Therefore,

$$\begin{aligned}
P(A \mid S_7) &= \frac{P(A \cap S_7)}{P(S_7)} = \frac{2/36}{6/36} = 1/3 \\
P(A \mid S_8) &= \frac{P(A \cap S_8)}{P(S_8)} = \frac{2/36}{5/36} = 2/5 \\
P(A \mid S_9) &= \frac{P(A \cap S_9)}{P(S_9)} = \frac{2/36}{4/36} = 1/2 \\
P(A \mid S_{10}) &= \frac{P(A \cap S_{10})}{P(S_{10})} = \frac{2/36}{3/36} = 2/3 \\
P(A \mid S_{11}) &= \frac{P(A \cap S_{11})}{P(S_{11})} = \frac{2/36}{2/36} = 1 \\
P(A \mid S_{12}) &= \frac{P(A \cap S_{12})}{P(S_{12})} = \frac{1/36}{1/36} = 1.
\end{aligned}$$

2. *The king comes from a family of 2 children. What is the probability his sibling is a girl?*

Solution: Let B_i denote the event that the ith child is a boy and $G_i = B_i^c$ the event that the ith child is a girl. Note that $G_1 \cup G_2$ is the event that there is at least one girl. Thus, we are asked to compute $P(G_1 \cup G_2 \mid B_1 \cup B_2)$. First of all,

$$P(B_1 \cup B_2) = 1 - P(G_1)P(G_2) = 3/4.$$

Also, note that $(G_1 \cup G_2) \cap (B_1 \cup B_2)$ is the event that we have a boy and a girl. Its probability is equal to $P(G_1 \cap B_2) + P(B_1 \cap G_2) = (1/4) + (1/4) = (1/2)$. Therefore, the answer is $\frac{1}{2}/\frac{3}{4} = (2/3)$.

3. *Suppose that 5% of all men and 0.25% of all women are colorblind. A colorblind person is chosen at random. What is the probability that this person is male? Assume that there are an equal number of men and women. What if the population consisted of twice as many mean as women?*

Solution: Let $M = \{\text{man}\}$ and $C = \{\text{color blind}\}$. We know: (a) $P(C \mid M) = 0.05$; (b) $P(C \mid M^c) = 0.0025$; (c) $P(M) = 1/2$. We seek to find $P(M \mid C)$. First, we reverse the order of conditioning:

$$P(M \mid C) = P(C \mid M)\frac{P(M)}{P(C)} = 0.05 \times \frac{0.5}{P(C)}.$$

Next, we use Bayes's formula to compute $P(C)$, viz.,

$$\begin{aligned} P(C) &= P(C \mid M)P(M) + P(C \mid M^c)P(M^c) \\ &= (0.05 \times 0.5) + (0.0025 \times 0.5) \approx 0.02625. \end{aligned}$$

Therefore, $P(M \mid C) \approx 0.9523$.

4. *Suppose we have 10 coins such that if the ith coin if flipped then heads will appear with probability $i/10$ ($i = 1, \ldots, 10$). One of the coins is selected at random, and then flipped. We are told that it shows heads. What is the probability that it was the fifth coin?*

Solution: Let C_i denote the event that the ith coin is selected. Then, we know that $P(H \mid C_i) = i/10$, and $P(C_i) = 1/10$. Therefore,

$$\begin{aligned} P(C_5 \mid H) &= \frac{P(C_5 \cap H)}{P(H)} \\ &= \frac{P(H \mid C_5)P(C_5)}{\sum_{i=1}^{10} P(H \mid C_i)P(C_i)} \\ &= \frac{(5/10) \times (1/10)}{\sum_{i=1}^{10}(i/10) \times (1/10)} \\ &= \frac{5}{1 + \cdots + 10} = \frac{1}{11}. \end{aligned}$$

The last line follows from the fact that $(1 + \cdots + 10) = \binom{11}{2} = 55$. Can you think of a combinatorial proof for this?) More generally, for all $\ell = 1, \ldots, 10$ we have

$$P(C_\ell \mid H) = \frac{\ell}{1 + \cdots + 10} = \frac{\ell}{55}.$$

5. *Let $S = \{1, \ldots, n\}$ and suppose that A and B are two independent, random subsets of S, all subsets being equally likely.*

(a) *Prove that $P\{A \subset B\} = (3/4)^n$.*

Solution: Let N_B denote the number of elements of B. Then, by Bayes' rule,

$$
\begin{aligned}
P\{A \subset B\} &= \sum_{k=0}^{n} P\left(A \subset B \mid N_B = k\right) \times P\{N_B = k\} \\
&= \sum_{k=0}^{n} \frac{2^k}{2^n} \times \frac{\binom{n}{k}}{2^n} = \frac{1}{4^n} \sum_{k=0}^{n} \binom{n}{k} 2^k \\
&= \frac{3^n}{4^n},
\end{aligned}
$$

by the binomial theorem.

(b) *Prove that $P\{A \cap B = \varnothing\} = (3/4)^n$.*

Solution: The event $\{A \cap B = \varnothing\}$ is the same as $\{A \subset B^c\}$. Because A and B^c are independent random subsets of S,

$$
\begin{aligned}
P\{A \subset B^c\} &= \sum_{k=0}^{n} P\left(A \subset B^c \mid N_{B^c} = k\right) \times P\{N_{B^c} = k\} \\
&= \sum_{k=0}^{n} \frac{2^k}{2^n} \times P\left\{N_B = n - k\right\} = \sum_{k=0}^{n} \frac{2^k}{2^n} \times \frac{\binom{n}{n-k}}{2^n} \\
&= \frac{1}{4^n} \sum_{k=0}^{n} \binom{n}{k} 2^k = \frac{3^n}{4^n}.
\end{aligned}
$$

Theoretical Problems:

1. *In Laplace's rule of succession (page 109, seventh edition), suppose that the first n flips resulted in r heads and $(n - r)$ tails. Show that the probability that the $(n + 1)$st flip turns up heads is $(r + 1)/(n + 2)$.*

Solution: Let E_n denote the event that, in the first n flips, there are r heads and $(n - r)$ tails. Let C_i denote the event that the ith coin was selected. We know that $P(C_i) = 1/(k + 1)$, and

$$
P(E_n \mid C_i) = \binom{n}{r} \left(\frac{i}{k}\right)^r \left(1 - \frac{i}{k}\right)^{n-r}.
$$

[Binomial probabilities.] Let H_i denote the event that, in the ith flip, we got heads. Then

$$\begin{aligned}
P(H_{n+1} \,|\, E_n) &= \sum_{i=0}^{k} P(H_{n+1} \cap C_i \,|\, E_n) \\
&= \sum_{i=0}^{k} P(H_{n+1} \,|\, C_i \cap E_n) P(C_i \,|\, E_n) \\
&= \sum_{i=0}^{k} \left(\frac{i}{k}\right) P(C_i \,|\, E_n),
\end{aligned}$$

since the $(n+1)$st toss is independent of the first n tosses. Now,

$$\begin{aligned}
P(C_i \,|\, E_n) &= P(E_n \,|\, C_i) \frac{P(C_i)}{P(E_n)} \\
&= \binom{n}{r} \left(\frac{i}{k}\right)^r \left(1 - \frac{i}{k}\right)^{n-r} \frac{1/(k+1)}{P(E_n)}.
\end{aligned}$$

By the Bayes's formula,

$$\begin{aligned}
P(E_n) &= \sum_{j=0}^{k} P(E_n \,|\, C_j) P(C_j) \\
&= \frac{1}{k+1} \sum_{j=0}^{k} \binom{n}{r} \left(\frac{j}{k}\right)^r \left(1 - \frac{j}{k}\right)^{n-r}.
\end{aligned}$$

Therefore, combine terms to find that

$$\begin{aligned}
P(H_{n+1} \,|\, E_n) &= \frac{\sum_{i=0}^{k} \left(\frac{i}{k}\right)^{r+1} \left(1 - \frac{i}{k}\right)^{n-r}}{\sum_{j=0}^{k} \left(\frac{j}{k}\right)^{r} \left(1 - \frac{j}{k}\right)^{n-r}} \\
&\to \frac{\int_0^1 x^{r+1}(1-x)^{n-r}\,dx}{\int_0^1 x^r(1-x)^{n-r}\,dx},
\end{aligned}$$

as $k \to \infty$. Suppose we could prove that

$$C(n,m) := \int_0^1 x^n (1-x)^m \, dx = \frac{n!m!}{(n+m+1)!}. \tag{1}$$

Then we have shown that

$$\begin{aligned}
\lim_{k\to\infty} P(H_{n+1} \,|\, E_n) &= \frac{C(r+1, n-r)}{C(r, n-r)} = \frac{(r+1)!(n-r)!/(n+2)!}{r!(n-r)!/(n+1)!} \\
&= \frac{r+1}{n+2},
\end{aligned}$$

This is (essentially) what we are asked to verify. It remains to prove (1). Now integrate by parts ($u = x^n, dv = (1-x)^m\,dx$) to find that $du = nx^{n-1}\,dx$, $v = -(1-x)^{m+1}/(m+1)$, and

$$\begin{aligned} C(n,m) &= \int_0^1 u\,dv = (uv)\Big|_{x=0}^{x=1} - \int_0^1 v\,du \\ &= \frac{n}{m+1}\int_0^1 x^{n-1}(1-x)^{m+1}\,dx = \frac{n}{m+1}C(n-1,m+1). \end{aligned}$$

Therefore,

$$\begin{aligned} C(n,m) &= \frac{n}{m+1}C(n-1,m+1) \\ &= \frac{n}{m+1}\cdot\frac{n-1}{m+2}C(n-2,m+2) \\ &\;\vdots \\ &= \frac{n}{m+1}\cdot\frac{n-1}{m+2}\cdots\frac{1}{m+n}C(0,m+n) \\ &= \frac{n}{m+1}\cdot\frac{n-1}{m+2}\cdots\frac{1}{m+n}\int_0^1 (1-x)^{n+m}\,dx \\ &= \frac{n}{m+1}\cdot\frac{n-1}{m+2}\cdots\frac{1}{m+n}\cdot\frac{1}{n+m+1}, \end{aligned}$$

as desired.