Lecture Notes on Multiparameter Processes: Ecole Polytechnique Fédérale de Lausanne, Switzerland

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Preface

These are the notes for a one-semester course based on ten lectures given at the *Ecole Polytechnique Fédérale de Lausanne*, April–June 2001. My goal has been to illustrate, in some detail, some of the salient features of the theory of multiparameter processes and in particular, Cairoli's theory of multiparameter martingales. In order to get to the heart of the matter, and develop a kind of intuition at the same time, I have chosen the simplest topics of random walks, Brownian motions, etc. to highlight the methods. The full theory can be found in *Multi-Parameter Processes: An Introduction to Random Fields* (henceforth, referred to as MPP) which is to be published by *Springer-Verlag*, although these lectures also contain material not covered in the mentioned book.

The first eight lectures are introductory material and everything is covered in some detail. The last two lectures are about potential theory of processes; in order to cover enough material, I have decided not to cover balayage; this makes the arguments of these two lectures somewhat heuristic, but the details are all there in MPP for those who are interested to learn more about the subject.

While I have tried to write the lectures faithfully, the heuristic "clumping" picture that I have tried to develop in the lectures is sadly missing from these notes. My hope is that, if all else is forgotten, one theme will be remembered, and that is the connections between our heuristic notion of clumping, and the rôle played by multiparameter martingales. Some of these notions are being rigorized in joint work with Yimin Xiao (under the general heading of *locally grown random sets*) as I write, but this will have to wait for now.

I am greatful to EPF-L for their hospitality, in particular, to Professor Robert C. Dalang, and for making my visit possible as well as highly enjoyable. Also, my heartfelt thanks go to all of the attendees of these lectures. Working with them has been a distinct pleasure for me. Last but not least, I wish to thank the United States' *National Science Foundation* for the generous support of my work on the topic of these lectures during the past six to seven years.

D. Khoshnevisan
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Lecture 1

Examples from Markov chains

In these lectures, I will develop some of the foundations of a theory of multiparameter Markov processes that is motivated by a number of problems coming from

- probability (intersections of processes, Brownian sheet, percolation on trees, Markov chains, potential theory);
- mathematical physics (Schrödinger operators, the $(\frac{1}{2}\Delta)^2$ operator of elasticity, polymer measures); as well as
- recent advances in geology (use of stable sheets in modeling rock strata instead of percolation models).

We will see that even in the study of classical stochastic processes, multiparameter processes arise as natural objects, although these connections sometimes go unnoticed.

Some of the material of this course is based on my forthcoming book *Multi-Parameter Processes: An Introduction to Random Fields* (Springer). I will make the relevant portions of this book available as needed during the lectures.

A rough outline of this course is as follows:

- Introduction and motivating examples from Markov chains, percolation, and Brownian motion;
- Capacity, energy, and Hausdorff dimension;
- Cairoli's theory of multiparameter martingales;
- Multiparameter Markov processes;
- The Fitzsimmons–Salisbury and the Hirsch–Song theories of potential;
- Brownian sheet and potential theory; Kahane's problem.

Time permitting, and depending on the audience's interests, we may discuss some aspects of

- Probability on sigma-finite spaces, Lévy processes and random walks; the Kesten–Bretagnolle criterion for Lévy processes ;
- analysis of sample paths, stochastic codimension.

Many of the techniques in these notes are based on various applications of the theory of martingales. In the one-parameter setting, this is Doob's theory, and in the multi-parameter setting, it is Cairoli's. Many of you may know that R. Cairoli was a Professor here at *EPF–Lausanne*, and his theory evolved here, locally.

In the first few lectures, we will play with some of the fundamental concepts developed later in this course, but in a simple setting, where it is easy to see what is going on, without need for theoretical developments.

1 Recurrence of Markov Chains

Let S be denumerable, and consider a Markov chain $X = \{X_n;\, n \geq 0\}$ on S. Recall that this means there are probabilities $\{\mathbb{P}_x;\, x \in S\}$, and *transition functions* $\{p_n;\, n \geq 0\}$, on $S \times S$, such that

- $\mathbb{P}_x\{X_0 = x\} = 1$, for all $x \in S$;
- $\mathbb{P}_x\{X_n = a \,|\, \mathcal{F}_k\} = \mathbb{P}_x\{X_n = a \,|\, X_k\} = p_{n-k}(X_k, a)$, $\mathbb{P}_x$-a.s. for all $n \geq k \geq 0$ and all $a \in S$, where $\mathcal{F} = \{\mathcal{F}_j;\, j \geq 0\}$ is the filtration generated by X.

One thinks of $p_n(a, b)$ as the probability that, in n time steps, the Markov chain goes from a to b.

Following Poincaré, Pólya, Chung, etc., we say that $x \in S$ is *recurrent*, if starting from $x \in S$, $n \mapsto X_n$ hits x infinitely often, with probability one. More precisely, x is recurrent if

$$\mathbb{P}_x\{\forall m \geq 0, \exists n \geq m :\ X_n = x\} = 1.$$

When is x recurrent? The classical condition of Pólya (for simple walks), Chung–Fuchs, etc. is

Theorem 1.1 (Pólya's Criterion) *$x \in S$ is recurrent if and only if $\sum_n p_n(x, x) = +\infty$.*

A rough explanation of the proof that is to come is needed. Namely, our proof uses the fact that if X hits x, it will do so several times in close proximity of one another. Thus, near the times when X hits x, one expects to observe an unusual contribution to the $\sum_j \mathbf{1}_{\{X_j = x\}}$. In even rougher terms, the random set $\{n :\ X_n = x\}$ is comprised of a bunch of i.i.d.–looking "clumps". The Paley–Zygmund inequality provides us with a key method in analysing clumping situations. It states

Lemma 1.2 (Paley–Zygmund's Inequality) *Suppose $Z \geq 0$ is a nontrivial random variable in $L^2(\mathbb{P})$. Then, for all $\varepsilon \in [0,1]$,*

$$\mathbb{P}\big(Z > \varepsilon \mathbb{E}\{Z\}\big) \geq (1-\varepsilon)^2 \frac{\big(\mathbb{E}\{Z\}\big)^2}{\mathbb{E}\{Z^2\}}.$$

Proof Using Cauchy–Schwarz inequality,

$$\begin{aligned}\mathbb{E}\{Z\} &\leq \varepsilon \mathbb{E}\{Z\} + \mathbb{E}\big(Z; Z > \varepsilon \mathbb{E}\{Z\}\big)\\ &\leq \varepsilon \mathbb{E}\{Z\} + \sqrt{\mathbb{E}\{Z^2\} \cdot \mathbb{P}\big(Z > \varepsilon \mathbb{E}\{Z\}\big)}.\end{aligned}$$

Just solve. $\square$

Proof of Theorem 1.1 Let $J_n = \sum_{j=0}^n \mathbf{1}_{\{X_j = x\}}$ to see that x is recurrent if and only if $\mathbb{P}_x\{J_\infty = \infty\} = 1$. Clearly,

$$\sum_n p_n(x,x) < +\infty \iff \mathbb{E}_x\{J_\infty\} < +\infty \implies J_\infty < +\infty, \ \mathbb{P}_x\text{-a.s.}$$

This is one half of the result. "Clumping" says that this should be sharp. To prove it, we assume that

$$\sum_n p_n(x,x) = \mathbb{E}_x\{J_\infty\} = +\infty,$$

and first notice that

$$\begin{aligned}\mathbb{E}_x\{J_n^2\} &\leq 2 \sum\sum_{0 \leq i \leq j \leq n} \mathbb{P}_x\{X_i = x\ ,\ X_j = x\}\\ &= 2 \sum\sum_{0 \leq i \leq j \leq n} p_i(x,x) p_{j-i}(x,x)\\ &\leq 2\big[\mathbb{E}_x\{J_n\}\big]^2.\end{aligned}$$

Of course, one always has $\|Z\|_2 \geq \|Z\|_1$. The above states that a kind of converse to this holds for $Z = J_n$.

This, together with the Paley–Zygmund inequality, shows for all $n \geq 1$ and all $\varepsilon \in [0,1]$,

$$\mathbb{P}_x\big(J_n \geq \varepsilon \mathbb{E}_x\{J_n\}\big) \geq \frac{(1-\varepsilon)^2}{2}.$$

Thus, using our assumption that $\mathbb{E}_x\{J_\infty\} = +\infty$, and choosing any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}_x\{J_\infty = +\infty\} &= \lim_{n\to\infty} \mathbb{P}_x\big(J_\infty \geq \varepsilon \mathbb{E}_x\{J_n\}\big)\\ &\geq \liminf_{n\to\infty} \mathbb{P}_x\big(J_n \geq \varepsilon \mathbb{E}_x\{J_n\}\big)\\ &\geq \tfrac{1}{2}(1-\varepsilon)^2.\end{aligned}$$

This being true for all $\varepsilon \in (0,1]$, we have shown that $\mathbb{P}_x\{J_\infty = +\infty\} \geq \frac{1}{2}$. We need to show this probability is 1. Usually, this step is done by stopping time arguments. Instead, we make an argument that has multiparameter extensions, and this is where martingales come in. Henceforth, we assume $\mathbb{P}_x\{J_\infty = \infty\}$ is positive, and strive to show that it is one.

Let us define $M_n = \mathbb{P}_x\{J_\infty = +\infty \,|\, \mathcal{F}_n\}$, and apply Doob's martingale convergence theorem. Indeed, $\mathcal{F}_n$ increases to $\vee_n \mathcal{F}_n$ which means that $\lim_n M_n = \mathbf{1}_{\{J_\infty = \infty\}}$, $\mathbb{P}_x$-a.s. On the other hand, by the Markov property, $M_n = \mathbb{P}_{X_n}\{J_\infty = +\infty\}$ (why?) Therefore, on the event that X_n hits x infinitely often (i.e., on $\{J_\infty = +\infty\}$), $M_n = \mathbb{P}_x\{J_\infty = +\infty\}$, infinitely often. In particular, $\mathbb{P}_x$-a.s. on $\{J_\infty = \infty\}$,

$$1 = \mathbf{1}_{\{J_\infty = \infty\}} = \lim_n M_n = \mathbb{P}_x\{J_\infty = \infty\}.$$

This finishes our proof. $\square$

SOMETHING TO TRY. Show that there are constants C_p such that for all $p > 1$ and all $n \geq 1$,

$$\|J_n\|_p \leq C_p \|J_n\|_1.$$

Thus, $\{J_n\}_{n \geq 1}$ forms what is called a uniformly hypercontractive family of operators on the underlying probability space. Hypercontractivity is known to be a powerful property in analysis.
(HINT. By Jensen's inequality, it suffices to do this for integers $p > 1$.)

2 Recurrence for Inhomogeneous Markov Chains

This subsection may not be in the lecture and can be omitted on first reading.

Recall that a process $\{X_n;\, n \geq 0\}$ is an *inhomogeneous Markov chain* if $\mathbb{P}_x\{X_n = a \,|\, \mathcal{F}_m\} = \mathbb{P}_x\{X_n = a \,|\, X_m\} = p_{m,n}(X_m, a)$, when $n > m$. Thus, the only difference between these and regular Markov chains is that the transition functions, $p_{m,n}$, need not satisfy $p_{m,n}(a,b) = p_{0,n-m}(a,b)$.

SOMETHING TO TRY. If X is an inhomogeneous Markov chain on S, the process $Y = \{(n, X_n);\, n \geq 0\}$ defines a homogeneous Markov chain on $\mathbb{Z} \times S$. The process Y is Doob's *space-time* process.

What happens when is x recurrent for an inhomogeneous Markov chain X? Perhaps not surprisingly, the answer is more complicated, although similar ideas as our proof of Theorem 1.1 still work. Namely, recurrence still occurs by clumping. However, due to the inhomogeneity of the chain, the clumps need not be evenly distributed. In more mathematical terms, we need to consider a weighted version of the number of times that the chain hits x. Due to the absence of 0–1 laws, we also revise our definition of recurrence and say that x is *recurrent* if $\mathbb{P}_x\{J_\infty = \infty\} > 0$, where J_∞ was defined in the previous subsection.

Supposing that F is a subset of $\mathbb{Z} = \{0, 1, \ldots, \}$, we write $\mu \in \mathcal{P}(F)$ meaning that $\{\mu_i;\, i \in F\}$ are probabilities: $\mu_i \in [0, 1]$ and $\sum_{i \in F} \mu_i = 1$. By the *energy* at x of such a μ, we mean the functional

$$\mathcal{E}_x(\mu) = \sum\sum_{\substack{0 \le k \le j \\ p_{0,j}(x,x) > 0}} \frac{p_{k,j}(x,x)}{p_{0,j}(x,x)}\, \mu_j\, \mu_k,$$

where, we recall, $p_{k,j}$'s are the transition functions of the inhomogeneous Markov chain X.

The *capacity* at x of F is defined by Gauss' minimum energy principle. Namely,

$$\mathcal{C}_x(F) = \Big[\inf_{\mu \in \mathcal{P}(F)} \mathcal{E}_x(\mu) \Big]^{-1},$$

where $\inf \varnothing = +\infty$, and $1 \div \infty = 0$. Then, we have the following quantitative estimate that is interesting even for homogeneous Markov chains:

Theorem 2.1 *For all $F \subseteq \mathbb{Z}$ and all $x \in S$,*

$$\tfrac{1}{2}\mathcal{C}_x(F \cap G_x) \le \mathbb{P}_x\{\exists n \in F :\, X_n = x\} \le \mathcal{C}_x(F \cap G_x),$$

where $G_x = \{i \ge 0 :\, \mathbb{P}_x[X_i = x] > 0\}$.
In particular, x is recurrent if and only if $\lim_{N \to \infty} \mathcal{C}_x(\{N, N+1, \ldots, \}) > 0$.

Lecture 2

Examples from Percolation on Trees and Brownian Motion

1 Proof of Theorem 2.1

The characterization of recurrence follows from the asserted inequalities, upon noticing that whenever $F \subset F'$, $\mathcal{C}(F) \le \mathcal{C}(F')$, so that $\lim_N \mathcal{C}(\{N, N+1, \ldots\})$ exists. It remains to verify the two inequalities.

For any $\mu \in \mathcal{P}(F \cap G_x)$, define

$$I_\mu = \sum_{i \in G_x} \frac{\mathbf{1}_{\{X_i = x\}}}{\mathbb{P}_x\{X_i = x\}} \, \mu_i.$$

Clearly, $\mathbb{P}_x\{X_i = x\} = p_{0,i}(x,x)$, and

$$\mathbb{E}_x\{I_\mu\} = 1. \tag{1.1}$$

This functional I_μ is a normalized, weighted, version of J_n in our proof of Theorem 1.1.1. With this in mind, we estimate its second moment in a similar way as we did that of J_n. Namely,

$$\begin{aligned} \mathbb{E}_x\{I_\mu^2\} &\le 2 \mathop{\sum\sum}_{\substack{i,j \in G_x: \\ i \le j}} \frac{\mathbb{P}_x\{X_i = X_j = x\}}{p_{0,i}(x,x) p_{0,j}(x,x)} \, \mu_i \, \mu_j \\ &= 2 \mathcal{E}_x(\mu), \end{aligned} \tag{1.2}$$

since $\mathbb{P}_x\{X_i = X_j = x\} = p_{0,i}(x,x) p_{i,j}(x,x)$ if $i \le j$. Eq. (1.1) and the above combine, thanks to the Paley–Zygmund inequality [Lemma 1.1.2], to show that for all $\mu \in \mathcal{P}(F \cap G_x)$,

$$\mathbb{P}_x\{\exists n \in F : \, X_n = x\} \ge \left[2 \mathcal{E}_x(\mu)\right]^{-1}.$$

Optimize over all $\mu \in \mathcal{P}(F)$ to derive the first inequality of Theorem 1.2.1. So far, we have followed our proof of Theorem 1.1.1 but with J_n replaced by I_μ. For the other bound, we need to do more work, since not all weights μ work.

For this bound, we can assume, without loss of generality, that $\mathbb{P}_x\{\exists n \in F : X_n = x\} > 0$. Let $T = \inf\{n \in F : X_n = x\}$, where $\inf \varnothing = +\infty$. Then, T is a stopping time and our assumption is equivalent to $\mathbb{P}_x\{T < \infty\} > 0$. In particular, $\mu \in \mathcal{P}(F \cap G_x)$, where we define

$$\mu_i = \mathbb{P}_x\{T = i \,|\, T < \infty\}, \qquad \forall i = 0, 1, 2, \ldots .$$

We consider the following martingale based on I_μ for *this* μ:

$$M_n = \mathbb{E}_x\{I_\mu \,|\, \mathcal{F}_n\}, \qquad \forall n \geq 0.$$

Clearly,

$$\begin{aligned} M_n &= \sum_{i \in G_x} \frac{\mathbb{P}_x\{X_i = x \,|\, \mathcal{F}_n\}}{p_{0,i}(x,x)}\, \mu_i \\ &\geq \sum_{\substack{i \in G_x: \\ i \geq n}} \frac{\mathbb{P}_x\{X_i = x \,|\, \mathcal{F}_n\}}{p_{0,i}(x,x)}\, \mu_i \\ &= \sum_{\substack{i \in G_x: \\ i \geq n}} \frac{p_{n,i}(X_n, x)}{p_{0,i}(x,x)}\, \mu_i \\ &\geq \sum_{\substack{i \in G_x: \\ i \geq n}} \frac{p_{n,i}(x, x)}{p_{0,i}(x,x)}\, \mu_i \cdot \mathbf{1}_{\{X_n = x\}}. \end{aligned}$$

In particular,

$$M_T \mathbf{1}_{\{T<\infty\}} \geq \sum_{\substack{i \in G_x: \\ i \geq T}} \frac{p_{T,i}(x,x)}{p_{0,i}(x,x)}\, \mu_i \cdot \mathbf{1}_{\{T<\infty\}}.$$

Taking expectations, and using the special form of μ gives

$$\mathbb{E}_x\{M_T; T < \infty\} \geq \mathcal{E}_x(\mu) \cdot \mathbb{P}_x\{T < \infty\}.$$

It remains to show that the left hand side is 1. But,

$$\begin{aligned} \mathbb{E}_x\{M_T; T < \infty\} &\leq \liminf_{n\to\infty} \mathbb{E}_x\{M_{T\wedge n}\} \\ &= \liminf_{n\to\infty} \mathbb{E}_x\{M_0\} \\ &= 1. \end{aligned}$$

Justification: the first inequality is from Fatou's lemma, the second equality holds by Doob's optional stopping theorem, and the third is from Eq. (1.1) above. This concludes our proof. $\square$

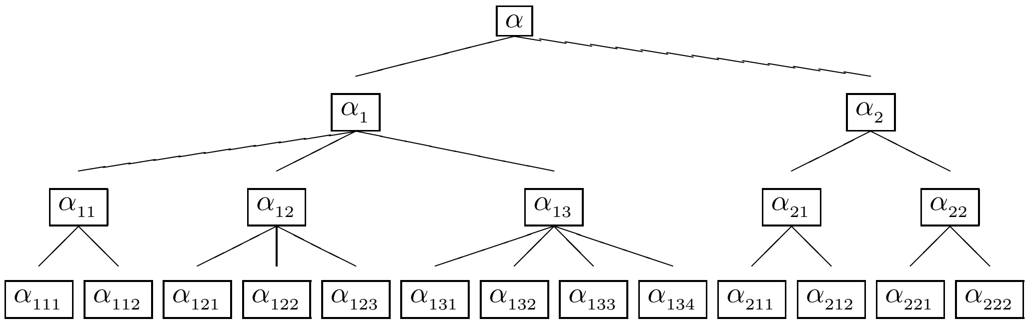

Figure 2.1: A typical ancestral tree Γ

2 Lyons' Theorem

Consider a finite rooted tree Γ, viewed as an ancestral tree, and with root α. An example of such a tree can be found in Figure 2.1, where α gives birth to two children, α_1 and α_2; α_1 gives birth to α_{11}, α_{12} and α_{13}, while α_2 gives birth to α_{21} and α_{22}, who give birth to $\cdots$.

The indices are chosen in a natural way, as is often done, to keep track of the ancestry of any given individual on the tree. In fact, this orders the vertices of Γ.

Every edge e in the tree is assigned a random weight, $w(e)$, and we suppose that $\{w(e); e \in \mathsf{Edge}(\Gamma)\}$ is an independent collection of random variables. In the percolation setting, $w(e) \in \{0, 1\}$; we can think of e as "*open*" when $w(e) = 1$, and "*closed*" when $w(e) = 0$.

We will assume that the weights $w(e)$ are i.i.d., to make our presentation simpler. Thus, there exists an inherent parameter, p, which is the probability $\mathsf{p} = \mathbb{P}\{w(e) = 1\}$. To emphasize the dependence on the parameter p, we write P_{p} for $\mathbb{P}$.

Now, we define percolation on Γ.

If β is some vertex in Γ, we write $\alpha \leftrightarrow \beta$ for the event that for all edges, e, that link α to β, $w(e) = 1$. If A is a collection of vertices, we write $\alpha \leftrightarrow A$ for the event $\cup_{\beta \in A}\{\alpha \leftrightarrow \beta\}$.

Let $\partial\Gamma$ denote the collection of all vertices in Γ whose graph distance from α is maximal. (In Figure 2.1, $\partial\Gamma = \{\alpha_{ijk}\}$ where the indices i, j and k range over the values $1 - 4$, as allowed by the the figure.) We then say that *percolation occurs on the finite tree* Γ, if $\alpha \leftrightarrow \partial\Gamma$.

Let us first look at a finite rooted tree Γ with root α, and let its depth be D.[1] [In Figure 2.1, $\mathsf{D} = 3$.] We can label, from left to right, the vertices in $\partial\Gamma$ to get vertices $1, \ldots, N$, where N is the number of vertices in $\partial\Gamma$. Thus, in Figure 2.1, we are relabeling $\alpha_{111}, \alpha_{112}, \ldots, \alpha_{221}, \alpha_{222}$ as vertices 1 through $N = 13$. For each of these N vertices, we can define a D-dimensional random vector, X_i, that is comprised of all the random weights, $w(e)$, for e's that link α to vertex i in $\partial\Gamma$. Then, it is easy to see that $X_1, \ldots, X_N$ is an inhomogeneous Markov chain, as long as we define $X_0 = x$ under the measure $\mathbb{P}_x$, where x takes its values in the space of all possible configurations of 1's and 0's corresponding to the rays from α to $\partial\Gamma$. In summary,

[1] Recall that D is the number of edges needed to go from the root to the boundary.

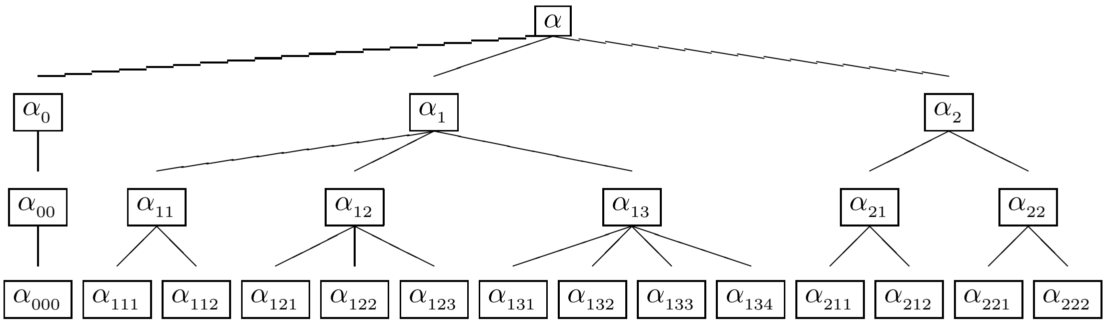

Figure 2.2: The ancestral tree of Figure 2.1, with an attached fictitious first ray

$x \in S = \{0,1\}^{\mathsf{D}}$. Pictorially, this means that for X to be an inhomogeneous Markov chain, we need to include an extra fictitious first ray that goes from α to $\partial\Gamma$ with no interference with the other rays.

Figure 2.2 shows what we do to Figure 2.1 in order to achieve this.

Now, it is easy to see that X is an inhomogeneous Markov chain on the state space $S = \{0,1\}^{\mathsf{D}}$.

We are interested in probability of percolation, which is

$$P_{\mathsf{p}}\{\alpha \leftrightarrow \partial\Gamma\} = \mathbb{P}_x\{\exists n \in \{1,\dots,N\} :\ X_n = x\},$$

where $x = \overbrace{(1,\dots,1)}^{\mathsf{D}\text{ times}}$.

Next, we compute the transition probabilities of this chain. Note that for $x = (1,\dots,1)$, as given, $p_{0,j}(x,x) = \mathsf{p}^{\mathsf{D}}$, since the above is just the probability that the ray leading to the j-th vertex on $\partial\Gamma$, counting from left, is all open. On the other hand, if $0 \le k < j$, $p_{k,j}(x,x)$ is the probability that both rays leading to k and j are all open. This probability is $p_{k,j}(x,x) = \mathsf{p}^{\mathsf{D}-|k \curlywedge j|}$, where $k \curlywedge j$ stands for the greatest common ancestor of k and j,[2] and where $|v|$ denotes the depth of any vertex v. In this way, we obtain

$$\frac{p_{0,j}(x,x)}{p_{k,j}(x,x)} = \mathsf{p}^{-|k \curlywedge j|}.$$

Since $G_x = \partial\Gamma$, we can combine all this with Theorem 2.1 to obtain

$$\left[2 \inf_{\mu \in \mathcal{P}(\partial\Gamma)} \sum\sum_{1 \le k \le j \le N} \mathsf{p}^{-|k \curlywedge j|}\mu_k\mu_j\right]^{-1} \le P_{\mathsf{p}}\{\alpha \leftrightarrow \partial\Gamma\} \le \left[\inf_{\mu \in \mathcal{P}(\partial\Gamma)} \sum\sum_{1 \le k \le j \le N} \mathsf{p}^{-|k \curlywedge j|}\mu_k\mu_j\right]^{-1}.$$

[2]This is a vertex on the tree, and is not to be mistaken with the number $k \wedge j$. For example, in Figure 2.1, $\alpha_{111} \curlywedge \alpha_{134} = \alpha_1$.

The notation $k \leq j$ is awkward, since it depends on a linear order of the bounadry of Γ, which makes no sense when we take Γ to be an infinite tree. So, it is more convenient to use the inequalities:

$$\tfrac{1}{2} \sum\sum_{1 \leq k \leq j \leq N} \leq \sum\sum_{1 \leq k \leq j \leq N} \leq \sum\sum_{1 \leq k \leq j \leq N},$$

to get

$$\Big[2 \inf_{\mu \in \mathcal{P}(\partial\Gamma)} \sum\sum_{k,j} \mathsf{p}^{-|k \curlywedge j|} \mu_k \mu_j\Big]^{-1} \leq P_{\mathsf{p}}\{\alpha \leftrightarrow \partial\Gamma\} \leq 2\Big[\inf_{\mu \in \mathcal{P}(\partial\Gamma)} \leq \sum\sum_{k,j} \mathsf{p}^{-|k \curlywedge j|} \mu_k \mu_j\Big]^{-1}.$$

When the tree is an infinite tree, the boundary $\partial\Gamma$ still makes sense: it is defined as the collection of all infinite rays emenating from α. It should be checked that when Γ is a finite tree, this agrees with our previous definition. Moreover, the notion of percolation still makes sense, as well. Namely, we write $\alpha \leftrightarrow \partial\Gamma$ to mean that for all rooted finite subtrees $\Gamma' \subset \Gamma$ with root α, $\alpha \leftrightarrow \partial\Gamma'$. It turns out that one can take "hydrodynamical limits", and let Γ become infinite. In this way, we get an improvement, due to Benjamini and Peres, of

Theorem 2.1 (Lyons) *Consider an infinite, locally finite, rooted tree Γ with root α as above. Then, for any $\mathsf{p} \in (0,1)$,*

$$\Big[2 \inf_{\mu \in \mathcal{P}(\partial\Gamma)} \iint \mathsf{p}^{-|\sigma \curlywedge \gamma|}\, \mu(d\sigma)\, \mu(d\gamma)\Big]^{-1} \leq P_{\mathsf{p}}\{\alpha \leftrightarrow \partial\Gamma\} \leq 2\Big[\inf_{\mu \in \mathcal{P}(\partial\Gamma)} \iint \mathsf{p}^{-|\sigma \curlywedge \gamma|}\, \mu(d\sigma)\, \mu(d\gamma)\Big]^{-1}.$$

In particular, there can be percolation iff there exists a probability measure μ, on $\partial\Gamma$, such that $\iint \mathsf{p}^{-|\gamma \curlywedge \sigma|}\, \mu(d\sigma)\, \mu(d\gamma) < +\infty$.

SOMETHING TO TRY: Check that the first inequality holds even if we remove the constant 2. (Hint: compute the first two moments of $I = \int \mathbf{1}_{\{\alpha \leftrightarrow \sigma\}} \div P_{\mathsf{P}}\{\alpha \leftrightarrow \sigma\}\, \mu(d\sigma)$, directly.

The above can be used to find critical percolation probabilities, as well as critical exponents, when the tree is regularly behaved. For example, if Γ is k-ary (or, more generally, radially symmetric), a μ that solves the optimization problem above is uniform measure on Γ, and this makes exact calculations possible; cf. Lyons (1990).

3 Brownian Motion

Next, we study another example of martingale techniques for clumping analysis by deriving a deep theorem of P. Lévy on the curve of a d-dimensional Brownian motion, $B = \{B(t);\, t \geq 0\}$, with $B(0) = 0$. Namely, we will show the following:

Theorem 3.1 (P. Lévy) *The random set $B(\mathbb{R}_+) \subset \mathbb{R}^d$ has positive d-dimensional Lebesgue's measure if and only if $d = 1$.*

When $d = 1$, this follows from the continuity of B, and from the fact that it is not a constant function. Thus, it suffices to reduce attention to $d \geq 2$ and show that for any interval $[L, U]$ where $0 < L < U < \infty$, $|B([L, U])| = 0$, almost surely. To see what this has to do with clumping, we first note that thank to Fubini's theorem,

$$\mathbb{E}\{|B([L, U])|\} = \int_{\mathbb{R}^d} \mathbb{P}\{a \in B([U, L])\} \, da, \tag{3.1}$$

where $| \bullet |$ refers to Lebesgue's measure. Thus, the question is "*when is* $\mathbb{P}\{a \in B([L, U])\} > 0$*?*" Now, it should not be surprising to see clumping at work: while $\mathbb{P}\{B(t) \approx a\}$ is small for any given t, $\mathbb{P}\{B(s) \approx a | B(t) \approx t\}$ is not. So, for $\{t \in [L, U] : B(t) = a\}$ to have any chance of being nonempty, it would have to be made up of clumps. Temporal homogeneity of B suggests that these clumps are, moreover, evenly distributed. All of this suggests looking at the random variable

$$J_\varepsilon(a) = \int_L^U \mathbf{1}_{\{|B(t)-a| \leq \varepsilon\}} \, ds, \tag{3.2}$$

where $a \in \mathbb{R}^d$ is fixed, and we think of ε as small.

Lecture 3

Proving Lévy's Theorem and Introducing Martingales

In this lecture, I will prove Theorem 3.1, and then, we will start our discussion of multiparameter martingales in earnest. For the latter part, I will pass out parts of my book, for the former $\cdots$ please read on.

1 Proof of Lévy's Theorem

Here is one estimate that follows easily from the form of the Gaussian density function.

Lemma 1.1 *For any $a \in \mathbb{R}^d$, there exists positive and finite constants $C_1 = C_1(a, U)$ and $C_2 = C_2(a, U)$, such that for all $s \in [0, U]$,*

$$C_1\Big(\frac{\varepsilon}{\sqrt{s}} \wedge 1\Big)^d \leq \mathbb{P}\{|B(s) - a| \leq \varepsilon\} \leq C_2\Big(\frac{\varepsilon}{\sqrt{s}} \wedge 1\Big)^d.$$

Consequently, since L is *strictly* positive, there exists nontrivial $C_3 = C_3(a, L, U)$ and $C_4 = C_4(a, L, U)$, such that for all sufficiently small $\varepsilon > 0$,

$$C_3\varepsilon^d \leq \mathbb{E}\{J_\varepsilon(a)\} \leq C_4\varepsilon^d. \tag{1.1}$$

Next, we estimate the second moment of $J_\varepsilon(a)$, viz.,

$$\begin{aligned}
\mathbb{E}\{|J_\varepsilon(a)|^2\} &= 2\int_L^U \int_s^U \mathbb{P}\{|B(s)-a| \le \varepsilon\,,\,|B(t)-a| \le \varepsilon\}\,dt\,ds \\
&\le 2\int_L^U \int_s^U \mathbb{P}\{|B(s)-a| \le \varepsilon\,,\,|B(t)-B(s)| \le 2\varepsilon\}\,dt\,ds \\
&\le 2\int_L^U \int_s^U \mathbb{P}\{|B(s)-a| \le \varepsilon\}\mathbb{P}\{|B(t-s)| \le 2\varepsilon\}\,dt\,ds \\
&\le 2\mathbb{E}\{J_\varepsilon(a)\} \cdot \int_0^U \mathbb{P}\{|B(u)| \le 2\varepsilon\}\,du \\
&\le 2C_2C_4\varepsilon^d \int_0^U \Big(\frac{2\varepsilon}{\sqrt{u}} \wedge 1\Big)^d\,du.
\end{aligned}$$

Here is where things get interesting. When $d \ge 3$, the behavior of the above integral is like a constant times ε^2. But, if $d=2$, it behaves like $\varepsilon^2\log(1/\varepsilon)$. Finally, if $d=1$, the integral behaves like ε (times a constant). All considered, we get

$$\mathbb{E}\{|J_\varepsilon(a)|^2\} \le C_5 \times \begin{cases} \varepsilon^2, & \text{if } d=1 \\ \varepsilon^2\log(1/\varepsilon), & \text{if } d=2 \\ \varepsilon^{d+2}, & \text{if } d \ge 3 \end{cases}.$$

Combine this with Eq. (1.1) and the Paley–Zygmund inequality [Lemma 1.1.2] to get

$$\begin{aligned}
\mathbb{P}\{\inf_{t\in[L,U]} |B(t)-a| \le \varepsilon\} &\ge \mathbb{P}\{J_\varepsilon(a) > 0\} \\
&\ge \frac{\big|\mathbb{E}\{J_\varepsilon(a)\}\big|^2}{\mathbb{E}\{|J_\varepsilon(a)|^2\}} \\
&\ge \frac{C_3^2}{C_5} \times \begin{cases} 1, & \text{if } d=1 \\ [\log(1/\varepsilon)]^{-1}, & \text{if } d=2 \\ \varepsilon^{d-2}, & \text{if } d \ge 3 \end{cases}.
\end{aligned} \tag{1.2}$$

I claim these bounds are sharp, up to multiplicative constants. Let $\mathcal{F}_t$ denote the filtration of B, and assume it satisfies the usual conditions. Let us consider

$$M_t = \mathbb{E}\Big\{\int_L^{L+U} \mathbf{1}_{\{|B(s)-a|\le 2\varepsilon\}}\,ds\,\Big|\,\mathcal{F}_t\Big\}.$$

This is almost the same as $\mathbb{E}\{J_{2\varepsilon}(a)|\mathcal{F}_t\}$, but we have increased the upper limit of integration for some elbow room; you will see why shortly. Clearly,

$$M_t \ge \int_t^{L+U} \mathbb{P}\{|B(s)-a| \le 2\varepsilon\,|\,\mathcal{F}_t\}\,ds, \qquad \forall t \in [L,U].$$

You may be wondering about the null sets. If so, that is good. However, there is a general fact about Brownian motion that states that $\mathbb{E}\{Z|\mathcal{F}_t\}$ can be chosen to be continuous. So, the above holds simultaneously for all $t \in [L, U]$, outside one null set. We go one more step:

$$\begin{aligned} M_t &\geq \int_t^{L+U} \mathbb{P}\{|B(s) - a| \leq 2\varepsilon \,|\, \mathcal{F}_t\}\, ds \times \mathbf{1}_{\{|B(t)-a| \leq \varepsilon\}}, && \forall t \in [L, U] \\ &\geq \int_t^{L+U} \mathbb{P}\{|B(s) - B(t)| \leq \varepsilon \,|\, \mathcal{F}_t\}\, ds \times \mathbf{1}_{\{|B(t)-a| \leq \varepsilon\}}, && \forall t \in [L, U] \\ &= \int_t^{L+U} \mathbb{P}\{|B(s-t)| \leq \varepsilon\}\, ds \times \mathbf{1}_{\{|B(t)-a| \leq \varepsilon\}}, && \forall t \in [L, U] \\ &\geq \int_0^{U} \mathbb{P}\{|B(u)| \leq \varepsilon\}\, du \times \mathbf{1}_{\{|B(t)-a| \leq \varepsilon\}}, && \forall t \in [L, U]. \end{aligned}$$

The point is that the above integral is nonrandom, and is an object we have seen before: when $d = 1$, it is of order ε^2, when $d = 2$, it is of order $\varepsilon^2 \log(1/\varepsilon)$, and when $d \geq 3$, it is of order ε^{d+2}. So, there must exist some constant $C_6 = C_6(a, d, U, L)$, such that

$$M_t \geq C_6 \mathbf{1}_{\{|B(t)-a| \leq \varepsilon\}} \times \begin{cases} \varepsilon, & \text{if } d = 1 \\ \varepsilon^2 \log(1/\varepsilon), & \text{if } d = 2 \\ \varepsilon^{d+2}, & \text{if } d \geq 3 \end{cases}, \qquad \forall t \in [L, U].$$

(We needed the extra "elbow room" to get $\forall t \in [L, U]$ in the above.) Now, let $T = \inf\{t \in [L, U] : |B(t) - a| \leq \varepsilon\}$ to see that

$$\mathbb{E}\{M_T \mathbf{1}_{\{T<\infty\}}\} \geq C_6 \mathbb{P}\{\inf_{t \in [L,U]} |B(t) - a| \leq \varepsilon\} \times \begin{cases} \varepsilon, & \text{if } d = 1 \\ \varepsilon^2 \log(1/\varepsilon), & \text{if } d = 2 \\ \varepsilon^{d+2}, & \text{if } d \geq 3 \end{cases}.$$

Thanks to the boundedness of M, and by Doob's optional stopping theorem, $\mathbb{E}\{M_T; T < \infty\} = \mathbb{E}\{J_\varepsilon(a)\}$, which is estimated by Eq. (1.1). Combining this and Eq. 1.2 we get

Theorem 1.2 *For any $a \in \mathbb{R}^d$, there exists constants A_1 and A_2, such that for all $\varepsilon > 0$ small,*

$$A_1\kappa(\varepsilon) \le \mathbb{P}\{\inf_{t\in[L,U]} |B(t) - a| \le \varepsilon\} \le A_2\kappa(\varepsilon),$$

where

$$\kappa(\varepsilon) = \begin{cases} 1, & \text{if } d = 1 \\ [\log(1/\varepsilon)]^{-1}, & \text{if } d = 2 \\ \varepsilon^{d-2}, & \text{if } d \ge 3 \end{cases}.$$

In particular, since B is a continuous random function, we can let $\varepsilon \to 0$ to see that when $d \ge 2$, $\mathbb{P}\{a \in B[L,U]\} = 0$ for all a, and we get Theorem 3.1 from integrating this $[da]$.

2 Review of Martingales

Recall that $\mathcal{F} = (\mathcal{F}_k)$ is a *filtration*, if $\mathcal{F}_k$'s are sigma-fields such that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$.

A stochastic process $X = (X_k)$ is *adapted* to the filtration $\mathcal{F}$, if for each k, X_k is $\mathcal{F}_k$-measurable.

A process $M = (M_k)$ is a *martingale*, if

- for each k, $M_k \in L^1(\mathbb{P})$;
- M is adapted; and
- for all k, $\mathbb{E}\{M_{k+1} \mid \mathcal{F}_k\} = M_k$, a.s.

Check that the third part is equivalent to: for all $n \ge m$, $\mathbb{E}\{M_m \mid \mathcal{F}_n\} = M_n$, a.s.

Proof We can assume that $n > m$. Otherwise, there is nothing to prove. Since $\mathcal{F}_n \supseteq \mathcal{F}_m$, $\mathbb{E}\{M_m \mid \mathcal{F}_n\} = \mathbb{E}\{\mathbb{E}\{M_m \mid \mathcal{F}_{m-1}\} \mid \mathcal{F}_n\} = \mathbb{E}\{M_{m-1} \mid \mathcal{F}_n\}$. Proceed by induction. □

Doob's Martingales are those of the form $M_k = \mathbb{E}\{Y \mid \mathcal{F}_k\}$, where $Y \in L^1(\mathbb{P})$.

A *stopping time* T is a $\mathbb{N} \cup \{\infty\}$-valued random variable such that for all k, $(T \le k) \in \mathcal{F}_k$.

The stopping time property always holds with respect to some filtration $\mathcal{F}$, of course.

If T is a stopping time, we define

$$\mathcal{F}_T = \big(A \in \vee_k \mathcal{F}_k : \ A \cap (T \le k) \in \mathcal{F}_k, \ \forall k\big).$$

Notes (i) $\mathcal{F}_T$ is a sigma-field; and (ii) both T and $X_T \mathbf{1}_{\{T<\infty\}}$ are $\mathcal{F}_T$-measurable.

Theorem 2.1 *Suppose $Y \in L^1(\mathbb{P})$ and $M_k = \mathbb{E}\{Y \,|\mathcal{F}_k\}$ for a given filtration $\mathcal{F}$. If T is a stopping time,*

$$M_T = \mathbb{E}\{Y \,|\mathcal{F}_T\}, \qquad \textit{on } (T < \infty).$$

Proof For all $A \in \mathcal{F}_T$,

$$\begin{aligned}
\mathbb{E}\{M_T; A \cap (T = k)\} &= \mathbb{E}\{M_k; A \cap (T = k)\} \\
&= \mathbb{E}\big\{\mathbb{E}(Y \,|\mathcal{F}_k); A \cap (T = k)\big\} \\
&= \mathbb{E}\big\{\mathbb{E}(Y; A \cap (T = k)|\mathcal{F}_k)\big\}, \qquad \text{since } A \cap (T = k) \in \mathcal{F}_k \\
&= \mathbb{E}\{Y; A \cap (T = k)\} \\
&= \mathbb{E}\big\{\mathbb{E}(Y \,|\mathcal{F}_T); A \cap (T = k)\big\}, \qquad \text{since } A \cap (T = k) \in \mathcal{F}_T.
\end{aligned}$$

Sum over all $k = 1, 2, \cdots$ to see that

$$\mathbb{E}\{M_T \mathbf{1}_{\{T<\infty\}}; A\} = \mathbb{E}\{\mathbb{E}(Y \,|\mathcal{F}_T)\mathbf{1}_{\{T<\infty\}}; A\}, \qquad \forall A \in \mathcal{F}_T.$$

Since $\mathbb{E}(Y \,|\mathcal{F}_T)\mathbf{1}_{\{T<\infty\}}$ and $M_T \mathbf{1}_{\{T<\infty\}}$ are both $\mathcal{F}_T$-measurable, this completes our proof; cf. "Notes" above for the latter remarks. $\square$

Theorem 2.2 (The Optional Stopping Theorem) *Suppose $T_1 \leq T_2$ are bounded stopping times, and M is a martingale, both with respect to the same filtration $\mathcal{F}$. Then,*

$$\mathbb{E}\{M_{T_2} |\mathcal{F}_{T_1}\} = M_{T_1}, \qquad \textit{a.s.}$$

Proof "Bounded" means that there exists a nonrandom $K > 0$, such that $T_1 \leq T_2 \leq K$. Note that

$$\forall j \leq K : \qquad M_j = \mathbb{E}\{M_{K+1} |\mathcal{F}_j\}.$$

Thus, by Theorem 2.1,

$$M_{T_2} = \sum_{j \leq K} M_j \mathbf{1}_{\{T_2 = j\}} = \mathbb{E}\{M_{K+1} |\mathcal{F}_{T_2}\}, \qquad \text{a.s.}$$

Since $\mathcal{F}_{T_2} \supseteq \mathcal{F}_{T_1}$, $\mathbb{E}\{M_{T_2} |\mathcal{F}_{T_1}\} = \mathbb{E}\{M_{K+1} |\mathcal{F}_{T_1}\}$, a.s. Another appeal to Theorem 2.1 does the job.

3 Doob's Maximal Inequalities

We can now state and prove Doob's martingale version of Kolmogorov's inequalities for random walks.

Theorem 3.1 *Suppose M is a martingale with respect to a filtration $\mathcal{F}$. Then, for all $\lambda > 0$,*

$$\mathbb{P}\{\max_{j \leq n} |M_j| \geq \lambda\} \leq \frac{1}{\lambda}\mathbb{E}\{|M_n|; \max_{j \leq n} |M_j| \geq \lambda\}.$$

Proof Let $T = \inf\{k : |M_k| \geq \lambda\}$ where $\inf \varnothing = \infty$. This is a stopping time, and $T \wedge n$ is a bounded stopping time.

By Theorem 2.2, $\mathbb{E}\{M_n | \mathcal{F}_{T\wedge n}\} = M_{T\wedge n}$. Applying Jensen's inequality, we deduce that a.s.,

$$\mathbb{E}\{|M_n| \,|\, \mathcal{F}_{T\wedge n}\} \geq |M_{T\wedge n}|.$$

Consequently,

$$\mathbb{E}\{|M_n|; T \leq n\} \geq \mathbb{E}\{|M_T|; T \leq n\} \geq \lambda \mathbb{P}\{T \leq n\},$$

giving the result, in a slightly different form. $\square$

Corollary 3.2 (Doob) *For all* $p > 1$,

$$\mathbb{E}\{\max_{j\leq n} |M_j|^p\} \leq \Big(\frac{p}{p-1}\Big)^p \mathbb{E}\{|M_n|^p\},$$

while

$$\mathbb{E}\{\max_{j\leq n} |M_j|\} \leq \Big(\frac{e}{e-1}\Big)\big\{1 + \mathbb{E}\{|M_n| \ln_+ |M_n|\}\big\},$$

where $\ln_+(x) = \ln(x \vee 1)$.

Proof We use integration by parts: for any random variable $Z \geq 0$, and for all $p \geq 1$,

$$\mathbb{E}\{Z^p\} = p\int_0^\infty \lambda^{p-1}\mathbb{P}\{Z > \lambda\}\, d\lambda.$$

Apply this to $Z = \max_{j\leq n} |M_j|$ to see that

$$\begin{aligned}
\mathbb{E}\{\max_{j\leq n} |M_j|^p\} &= p\int_0^\infty \lambda^{p-1}\mathbb{P}\{\max_{j\leq n} |M_j| > \lambda\}\, d\lambda \\
&\leq p\int_0^\infty \lambda^{p-2}\mathbb{E}\{|M_n|; \lambda \leq \max_{j\leq n} |M_j|\}\, d\lambda \\
&= p\mathbb{E}\Big\{|M_n| \cdot \int_0^{\max_{j\leq n}|M_j|} \lambda^{p-2}\, d\lambda\Big\} \qquad \text{(Fubini' theorem)} \\
&= \Big(\frac{p}{p-1}\Big)\mathbb{E}\{|M_n| \cdot \max_{j\leq n} |M_j|^{p-1}\}.
\end{aligned}$$

By Hölder's inequality, $\mathbb{E}\{|M_n| \cdot \max_{j\leq n} |M_j|^{p-1}\} \leq \|M_n\|_p \cdot \|\max_{j\leq n} |M_j|\|_p^{p-1}$. The result follows from this when $p > 1$. For the $p = 1$ case, see Chapter 1 of MPP. $\square$

Lecture 4

Preliminaries on Ortho-Martingales

Before our general discussion of ortho-martingales, let us look at a simpler, more concrete, class of multi-parameter "martingales"; these first arose in Cairoli (1969), and have resurfaced in many works, including those of Cairoli and Walsh, several times since, and for a number of different reasons.

1 Two Parameters Doob-Type Ortho-Martingales

Suppose $\mathcal{F}^1$ and $\mathcal{F}^2$ are two one-parameter filtrations. To be sure that we are getting the indices right, we stress that this notation means, in particular, that $\mathcal{F}^1 = (\mathcal{F}^1_1, \mathcal{F}^1_2 \ldots)$ and $\mathcal{F}^2 = (\mathcal{F}^2_1, \mathcal{F}^2_2, \ldots)$ are two ordinary filtrations on the same underlying probability space.

By a *two-parameter Doob-type ortho-martingale,* we mean the process

$$M_{n,m} = \mathbb{E}\{Y \mid \mathcal{F}^1_n \vee \mathcal{F}^2_m\}, \qquad \forall n, m \geq 1,$$

where $Y \in L^1(\mathbb{P})$.

We should recognize that

$$\begin{aligned} M_{n,m} &= \mathbb{E}\{M_{n+1,m} \mid \mathcal{F}^1_n \vee \mathcal{F}^2_m\} \\ &= \mathbb{E}\{M_{n,m+1} \mid \mathcal{F}^1_n \vee \mathcal{F}^2_m\} \\ &= \mathbb{E}\{M_{n+i,m+j} \mid \mathcal{F}^1_n \vee \mathcal{F}^2_m\}, \end{aligned} \tag{1.1}$$

for *any* $i, j \geq 0$. This is why we call M a *Doob-type ortho-martingale,* since an ortho-martingale is to be thought of as an *orthant-wise martingale,* i.e., a martingale in each parameter.

Lemma 1.1 *For all $p > 1$, and for all integers $n, m \geq 1$,*

$$\mathbb{E}\Big\{ \max_{\substack{i\leq n \\ j\leq m}} |M_{i,j}|^p \Big\} \leq \left(\frac{p}{p-1}\right)^{2p} \mathbb{E}\{|Y|^p\}.$$

Proof If we fix i, $j \mapsto M_{i,j}$ is a 1-parameter Doob martingale; cf. Eq. (1.1). Therefore, by Doob's inequality (Corollary 3.2, Lecture 3),

$$\mathbb{E}\{\max_{j\leq m} |M_{i,j}|^p\} \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\{|M_{i,m}|^p\}, \qquad \forall i \geq 1.$$

Another application of Eq. (1.1) reveals that $i \mapsto \max_{j\leq m} |M_{i,j}|^p$ is a 1-parameter submartingale (that I have not defined in the lectures, but you know about.) Therefore, we can apply Doob's inequality again to see that

$$\mathbb{E}\{\max_{i\leq n} \max_{j\leq m} |M_{i,j}|^p\} \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\{\max_{j\leq m} |M_{n,m}|^p\},$$

which, together with the previous display, proves

$$\mathbb{E}\{\max_{i\leq n} \max_{j\leq m} |M_{i,j}|^p\} \leq \left(\frac{p}{p-1}\right)^{2p} \{|M_{n,m}|\},$$

which has the desired result, thanks to Jensen's inequality. □

What about the $p = 1$ case?

Lemma 1.2 *For all $n, m \geq 1$,*

$$\mathbb{E}\{\max_{\substack{i\leq n \\ j\leq m}} |M_{i,j}|\} \leq \left(\frac{e}{e-1}\right)^2 \Big[2 + \mathbb{E}\big\{ |M_{n,m}| \ln_+ |M_{n,m}|\big\}\Big].$$

For a proof, see Ch. 1 of MPP.

Finally, we have

Lemma 1.3 *For all $n, m \geq 1$ and all $\lambda > 0$,*

$$\mathbb{P}\{\max_{\substack{i\leq n\\ j\leq m}} |M_{i,j}| \geq \lambda\} \leq \frac{1}{\lambda}\Big(\frac{e}{e-1}\Big)\Big[1 + \mathbb{E}\big\{|M_{n,m}| \ln_+ |M_{n,m}|\big\}\Big].$$

Proof Since $\max_{j\leq m} |M_{i,j}|$ is a submartingale for each $i \geq 1$, by Doob's maximal inequality,

$$\mathbb{P}\{\max_{\substack{i\leq n\\ j\leq m}} |M_{i,j}| \geq \lambda\} \leq \frac{1}{\lambda}\mathbb{E}\{\max_{j\leq m} |M_{n,j}|\},$$

and another application of Doob's maximal inequality does the job; cf. (Corollary 3.2, Lecture 3). $\square$

2 Two-Parameter Convergence Theorems

Continuing with our discussion of the previous section, we now address the question of when, if ever, $\lim M_{n,m}$ exists.

One can imagine many different notions of limits in the present 2-parameter setting. Here are two important ones:

- **Topological limits.** We say that M has a *topological limit* (at infinity) if with probability one, $\lim_{n,m\to\infty} M_{n,m}$ exists. That is, outside one null set, we can let n and m go to infinity in any way we like and still $M_{n,m}$ converges. This is, indeed, a topological limit, as can be argued simply as follows: Consider $\mathbb{R}^2/\partial\mathbb{R}^2$, i.e., $\mathbb{R}^2$ where we identify all points on the axes with each other. Endow it with the relative topology, and then, one-point compactify it to see that $\lim_{n,m\to\infty}$ means (n, m) converges, in the latter topology, to the added compactification points ∞, say.

- **Pathwise limits.** In contrast with topological convergence, pathwise convergence is a probabilistic notion. A collection of points $(i_1, j_1), \ldots, (i_k, j_k)$ is an *increasing path* if $i_1 \leq \cdots \leq i_k$ and $j_1 \leq \cdots \leq j_k$. This extends to the case where some of the i_ℓ's and j_ℓ's are infinity, as well. With this in mind, we say that our two-parameter orthomartingale M has *pathwise limits* if for every increasing path $(i_1, j_1), (i_2, j_2), \ldots$ there exists a null set outside which $\lim_{\ell\to\infty} M_{i_\ell,j_\ell}$ exists. Both the limit, here, as well as the null set may depend on the increasing path.

Remark 1 Check that any real-valued function $f(n, m)$ has topological limits iff $\lim_n f(n, m)$ (and $\lim_m f(n, m)$, resp.) exists *uniformly* in m (in n, resp.)

Remark 2 If M is non random, the existence of a topological limit at infinity is equivalent to the existence of pathwise limits at infinity (i.e., when the increasing paths go away from the axes in all directions.) For nonrandom processes, this is not true, in general, since there are uncountably many increasing paths.

Theorem 2.1 *The process M always has limits. Moreover, its pathwise limit at infinity is $\mathbb{E}\{Y \mid \mathcal{F}^1_\infty \vee \mathcal{F}^2_\infty\}$, where $\mathcal{F}^\ell_\infty = \vee_{j\geq 1}\mathcal{F}^\ell_j$. Finally, it has a topological limit at infinity if $\mathbb{E}\{|Y| \ln_+ |Y|\} < \infty$.*

Proof The existence of pathwise limits is easy: let $(i_1, j_1), (i_2, j_2), \ldots$ denote any increasing path to deduce that $\ell \mapsto M_{i_\ell, j_\ell}$ is a uniformly integrable, one-parameter martingale with respect to the one-parameter filtration $\mathcal{F}^1_{i_\ell} \vee \mathcal{F}^2_{j_\ell}$. Thus, by Doob's theorem, it converges to $\mathbb{E}\{Y \mid \mathcal{F}^1_\infty \vee \mathcal{F}^2_\infty\}$, a.s. in and $L^1(\mathbb{P})$.

Next, we suppose $Y \in L \ln_+ L$, and aim to prove topological convergence. Let $\Psi(x) = x \ln_+(x)$ ($x > 0$) and note that

$$\Psi(|x - y|) \leq |\Psi(x) - \Psi(y)|, \qquad \forall x, y > 0.$$

This requires a few lines of calculations, but also follows from convexity consideration; cf. Ch. 1 of MPP.

The preceeding discussion, together with Lebesgue's dominated convergence theorem, imply that when $Y \in L \ln_+ L$,

$$\lim_{i,j\to\infty} \mathbb{E}\{\Psi(|M_{i,j} - Y|)\} = 0.$$

In particular, there exists a sequence of constants, $c_1 \leq c_2 \leq \ldots$, such that $\lim_{j\to\infty} c_j = \infty$, and

$$\mathbb{E}\{\Psi(c_j |M_{i,j} - Y|)\} \leq 1.$$

(Why?) Now, $c_j\{M_{i,j} - Y\}$ is also a two-parameter Doob-type ortho-martingale. Thus, we apply Lemma 1.3 to the latter and deduce that for all $\lambda > 0$,

$$\mathbb{P}\{\sup_{\substack{i\geq 1\\ j\geq m}} |M_{i,j} - Y| \geq \lambda\} \leq \frac{1}{c_m \lambda} \sup_{j\geq m} \sup_{i\geq 1} \mathbb{E}\{\Psi(c_j |M_{i,j} - Y|)\} \leq \frac{1}{c_m \lambda}.$$

The above goes to zero, as $m \to \infty$, and this is enough to show that for all $\lambda > 0$,

$$\limsup_{m\to\infty} \sup_{\substack{i\geq 1\\ j\geq m}} |M_{i,j} - Y| < \lambda, \qquad \text{a.s.}$$

Since the above holds for all $\lambda > 0$, this proves our result. $\square$

3 Further Discussion

In general, we have N one-parameter filtrations, $\mathcal{F}^1, \ldots, \mathcal{F}^N$ and say that an N-parameter process is an *ortho-martingale* (with respect to these filtrations) if

- M_t is $\mathcal{F}^1_{t_1} \vee \cdots \vee \mathcal{F}^N_{t_N}$-measurable for all $t \in \mathbb{N}^N$;
- for all $t \in \mathbb{N}^N$, $M_t \in L^1(\mathbb{P})$; and
- whenever $t_i \geq s_i$ for all $i = 1, \ldots, N$ (written $t \succcurlyeq s$),

$$\mathbb{E}\{M_t \,|\, \mathcal{F}^1_{s_1} \vee \cdots \vee \mathcal{F}^N_{s_N}\} = M_s, \qquad \text{a.s.}$$

We, then, have the following, whose proof is based on ideas that we have already seen in the simpler context of two-parameter Doob-type ortho-martingales.

Theorem 3.1 (Cairoli) ***Maximal Inequalities:*** *Let M be an N-parameter ortho-martingale. Then,*

$$\mathbb{E}\big\{\max_{s \preccurlyeq t} |M_s|^p\big\} \leq \Big(\frac{p}{p-1}\Big)^{Np} \mathbb{E}\{|M_t|^p\},$$

if $p > 1$. If $p = 1$,

$$\mathbb{E}\big\{\max_{s \preccurlyeq t} |M_s|\big\} \leq \Big(\frac{e}{e-1}\Big)^N \Big[N + \mathbb{E}\big\{|M_t|[\ln_+ |M_t|]^N\big\}\Big].$$

Convergence Theorems *If M_t is uniformly integrable for all $t \in \mathbb{N}^N$, M has pathwise limits a.s. and in $L^1(\mathbb{P})$. Moreover, its pathwise limit at infinity is $\mathbb{E}\{Y \,|\mathcal{F}^1_\infty \vee \cdots \mathcal{F}^N_\infty\}$. If $\sup_t \mathbb{E}\{|M_t|[\ln_+ |M_t|]^{N-1}\} < +\infty$, M has topological limits, a.s. and in $L^1(\mathbb{P})$.*

4 Random Walk Examples

4.1 Additive Random Walks

Suppose X and Y are two independent mean 0 random walks, and define the *additive random walk*, S, by

$$S_{n,m} = X_n + Y_m, \qquad \forall n, m \geq 1.$$

Let $\mathcal{F}^1$ and $\mathcal{F}^2$ denote the filtrations generated by X and Y, respectively. That is,

$$\mathcal{F}^1_i = \sigma(X_\ell;\, \ell \leq i), \text{ and}$$
$$\mathcal{F}^2_j = \sigma(Y_\ell;\, \ell \leq j).$$

Then, it is easy to see that S is a two-parameter ortho-martingale.

4.2 Multi-Parameter Random Walks

Suppose $X_{i,j}$ are i.i.d. mean zero random variables. The *random walk* associated with the X's is

$$S_{n,m} = \sum_{\substack{i \le n \\ j \le m}} X_{i,j}, \qquad \forall n, m \ge 1.$$

Now, we need some filtrations. Let

$$\mathcal{F}_i^1 = \sigma(X_{\ell,j};\, \ell \le i, j \ge 1)$$
$$\mathcal{F}_j^2 = \sigma(X_{i,\ell};\, i \ge 1, \ell \le j).$$

Then, it is not hard, and very instructive, to check that for any $n, m \ge 1$ *fixed*: $\{S_{i,j}; i \le n,\ j \le m\}$ is a two-parameter Doob-type ortho-martingale with respect to these filtrations. In fact, $\frac{1}{nm} S_{n,m}$ is a reversed "ortho-martingale" (later) with respect to the reversed filtrations

$$\mathcal{R}_i^1 = \sigma(X_{\ell,j};\, \ell \ge i, j \ge 1)$$
$$\mathcal{R}_j^2 = \sigma(X_{i,\ell};\, i \ge 1, \ell \ge j).$$

This, and the reversed analogue of Theorem 3.1 for ortho-martingales, together, prove the interesting half of *Smythe's law of large numbers*:

Theorem 4.1 (Smythe) *If $X_{1,1} \in L \ln_+ L$,*

$$\lim_{n,m \to \infty} \frac{S_{n,m}}{nm} = \mathbb{E}\{X_{1,1}\}, \qquad \textit{a.s. and in } L^1(\mathbb{P}).$$

Moreover, if $(n,m) \mapsto \frac{1}{nm} S_{n,m}$ has topological limits at infinity, then $X_{1,1} \in L \ln_+ L$.

See MPP and Cairoli and Dalang (1996) for complete pedagogical proofs, as well as related results .

Lecture 5

Ortho-Martingales and Intersections of Walks and Brownian Motion

Having motivated orthomartingales somewhat, we proceed in earnest, cookbook style.

1 Ortho-martingales: The Recipe

Ingredients

1. An N-parameter process $\{M_t;\ t \in \mathbb{N}^N\}$;
2. N one-parameter filtrations $\mathcal{F}^1, \ldots, \mathcal{F}^N$, such that for each $i = 1, \ldots, N$, $t^{(i)} \mapsto M_t$ is a martingale with respect to $\mathcal{F}^i$.

For instance, take $N = 2$: then M is a martingale (with respect to $\mathcal{F}^1$ and $\mathcal{F}^2$), if

- for all j, $i \mapsto M_{i,j}$ is a martingale for $\mathcal{F}^1$; and
- for all i, $j \mapsto M_{i,j}$ is a martingale for $\mathcal{F}^2$.

Example (revisited) Let X^1 and X^2 be two *independent* mean zero random walks, and consider

$$\begin{aligned}\mathcal{F}^1_i &= \sigma(X^1_1, \ldots, X^1_i) \vee \sigma(X^2_1, X^2_2, \ldots)\\ \mathcal{F}^2_j &= \sigma(X^1_1, X^1_2, \ldots) \vee \sigma(X^2_1, \ldots, X^2_j).\end{aligned}$$

Then, $(n, m) \mapsto X^1_n + X^2_m$ is a two-parameter ortho-martingale for $\mathcal{F}^1$ and $\mathcal{F}^2$.

SOMETHING TO TRY: Show that $(n, m) \mapsto X_n^1 \cdot X_m^2$ is an ortho-martingale if X^1 and X^2 are independent martingales.

Example (revisited) Suppose $X_{i,j}$ are i.i.d. mean zero random variables and define the 2-parameter random walk S by

$$S_{n,m} = \sum_{i \le n} \sum_{j \le m} X_{i,j}.$$

Then, S is a 2-parameter ortho-martingale with respect to

$$\begin{aligned}\mathcal{F}_i^1 &= \sigma(X_{\ell,j} : \ \ell \le i, \ j \ge 1)\\ \mathcal{F}_j^2 &= \sigma(X_{i,\ell} : \ i \ge 1, \ \ell \le j).\end{aligned}$$

Henceforth, when $s, t \in \mathbb{R}^N$ and we write $s \preccurlyeq t$, we mean $s^{(i)} \le t^{(i)}$ for each coordinate $i = 1, \ldots, N$.

SOMETHING TO TRY: Check that when M is an ortho-martingale and when $s \preccurlyeq t$ are both in $\mathbb{N}^N$, then $\mathbb{E}\{|M_s|^p\} \le \mathbb{E}\{|M_t|^p\}$ for $p > 1$. (Hint: Jensen's inequality.)

Now, we come to ourn maximal inequalities; they are proved working one parameter at a time, just as the Doob-type examples that were worked out earlier.

Theorem 1.1 (Cairoli's Inequalities) *If M is an N-parameter ortho-martingale indexed by $\mathbb{N}^N$ say. Then,*

$$\begin{aligned}\mathbb{E}\{\max_{s \preccurlyeq t} |M_s|^p\} &\le C_{p,N} \mathbb{E}\{|M_t|^p\}\\ \mathbb{E}\{\max_{s \preccurlyeq t} |M_s|\} &\le C_{1,N} \Big[N + \mathbb{E}\{|M_t| \ln_+^N |M_t|\}\Big]\\ \mathbb{P}\{\max_{s \preccurlyeq t} |M_s| \ge \lambda\} &\le \frac{C_{1,N-1}}{\lambda} \Big[(N-1) + \mathbb{E}\{|M_t| \ln^{N-1} |M_t|\}\Big], \qquad \forall \lambda > 0,\end{aligned}$$

where

$$C_{p,N} = \left(\frac{p}{p-1}\right)^{Np} \qquad \textit{and} \qquad C_{1,N} = \left(\frac{e}{e-1}\right)^N.$$

There is a topological convergence theorem, as well. It is proved by working one parameter at a time, as in our example from last week, viz.,

Theorem 1.2 *If M is an ortho-martingale, and if $\sup_t \mathbb{E}\{|M_t| \ln_+^{N-1} |M_t|\} < +\infty$, then the topological limit $\lim_{t\to\infty} M_t$ exists, a.s. and in $L^1(\mathbb{P})$.*

The analogue of pathwise limits, at this level of generality, is tricky. Instead, we will introduce sectorial limits when $N = 2$. The remaining details can be found in MPP.

Henceforth, $N = 2$, in this section.

The two **sectorial limits**, when they exist, of a two-variable function f are defined by the ordered limiting operations: $\lim_{n\to\infty} \lim_{m\to\infty} f(n,m)$ and $\lim_{m\to\infty} \lim_{n\to\infty} f(n,m)$. In general, even when they exist, they need not be equal. When both sectorial limits exist and when they both agree, we write $\lim_{n,m\rightsquigarrow\infty} f(n,m)$ for their common value, and call it **the sectorial limit** of f.

Theorem 1.3 *If M is a uniformly integrable ortho-martingale, $\lim_{t\rightsquigarrow\infty} M_t$ exists a.s. and in $L^2(\mathbb{P})$.*

Proof for $N = 2$ By uniform integrability, $M_{i,\infty} = \lim_{j\to\infty} M_{i,j}$ exists a.s. and in $L^1(\mathbb{P})$. We now make two claims:

Claim 1 $i \mapsto M_{i,\infty}$ is uniformly integrable.

Claim 2 $i \mapsto M_{i,\infty}$ is an $\mathcal{F}^1$-submartingale.

To prove Claim 1, note that by Fatou's lemma, for all $\lambda > 0$,

$$\mathbb{E}\{|M_{i,\infty}|; |M_{i,\infty}| \geq \lambda\} \leq \lim_{j\to\infty} \mathbb{E}\{|M_{i,j}|; |M_{i,j}| \geq \lambda\} \leq \sup_j \mathbb{E}\{|M_{i,j}|; |M_{i,j}| \geq \lambda\},$$

which goes to 0 as $\lambda \to \infty$, uniformly in i. Claim 2 is proved similarly. Indeed,

$$\mathbb{E}\{M_{i+1,\infty} \,|\, \mathcal{F}_i^1\} \leq \liminf_{j\to\infty} \mathbb{E}\{M_{i+1,j} \,|\, \mathcal{F}_i^1\} = \lim_{j\to\infty} M_{i,j} = M_{i,\infty}.$$

In particular, $M^1_{\infty,\infty} = \lim_{i\to\infty} M_{i,\infty}$ exists. Of course, $M^1_{\infty,\infty} = \lim_i \lim_j M_{i,j}$ is the first sectorial limit, which we now know exists. Similarly, $M_{\infty,j} = \lim_i M_{i,j}$ exists a.s. and in $L^1(\mathbb{P})$, which leads to the existence of the second sectorial limit, $M^2_{\infty,\infty} = \lim_j \lim_i M_{i,j}$.

Our job, now, is to check that $M^1_{\infty,\infty} = M^2_{\infty,\infty}$. But for any $k \geq 1$,

$$\begin{aligned}
\mathbb{E}\{M^1_{\infty,\infty} \,|\, \mathcal{F}^1_k\} &\leq \liminf_{i\to\infty} \liminf_{j\to\infty} \mathbb{E}\{M_{i,j} \,|\, \mathcal{F}^1_k\} \\
&= \lim_{i\to\infty} \lim_{j\to\infty} M_{k,j} \qquad \text{(by the ortho-martingale property)} \\
&= M_{k,\infty}.
\end{aligned}$$

Let $k \to \infty$ to see that

$$\mathbb{E}\{M^1_{\infty,\infty} \mid \vee_k \mathcal{F}^1_k\} \leq M^2_{\infty,\infty}.$$

But $M^1_{\infty,\infty}$ is always $\vee_k \mathcal{F}^1_k$-measurable. So, $M^1_{\infty,\infty} \leq M^2_{\infty,\infty}$, which, by symmetry, shows that the two sectorial limits agree, a.s. $\square$

2 Applications to Intersections of Simple Walks

Our next goal is to apply the maximal inequality for ortho-martingales to a result about the intersections of independent Brownian motions. To illustrate some of the moment estimates, we first turn to a discrete version, which is simple to work out, although it does not really need ortho-martingale theory.

Let X^1 and X^2 be two simple symmetric random walks in $\mathbb{Z}^d$, and consider

$$J_{n,m} = \sum_{i\leq n} \sum_{j\leq m} \mathbf{1}\{X^1_i = X^2_j\},$$

which is nothing other than the number of times the trajectories of the random functions X^1 and X^2 intersect.

How big is $J_{n,n}$? Its mean is easy to estimate. Recall, from the classical local limit theorem, that if S_n is the simple walk in $\mathbb{Z}^d$,

$$\begin{aligned}
\mathbb{P}\{S_{2n} = 0\} &\sim C_d (2n)^{-\frac{d}{2}}, \qquad \text{as } n \to \infty, \text{ and} \\
\sup_{a\in\mathbb{Z}^d} \mathbb{P}\{S_k = a\} &\leq C_d k^{-\frac{d}{2}}, \qquad \forall k \geq 1.
\end{aligned} \tag{2.1}$$

Of course, $\mathbb{P}\{S_{2n+1} = 0\} = 0$, for all n, and $a_n \sim b_n$ means that $\lim_n a_n \div b_n = 1$.

Now, we can estimate $\mathbb{E}\{J_{n,m}\}$, since $X^1_i + X^2_j$ has the same distribution as S_{i+j}. Thus,

$$\mathbb{E}\{J_{n,n}\} \sim C'_d \sum_{i\leq n} \sum_{j\leq n} (i+j)^{-\frac{d}{2}},$$

as $n \to \infty$. The above has the same asymptotics as $\int_1^n \int_1^n (x+y)^{-\frac{d}{2}}\, dx\, dy$ which can be computed explicitly. However, it is more elegant to observe that

$$\begin{aligned}\int_{[1,n]^2} (x+y)^{-\frac{d}{2}}\, dx\, dy &\asymp \int_{\substack{z\in\mathbb{R}^2:\\ 1\le\|z\|\le cn}} \|z\|^{-\frac{d}{2}}\, dz \\ &= C\int_1^{cn} w^{-\frac{d}{2}+1}\, dw,\end{aligned}$$

where $\asymp$ means 'has the same rough asymptotics as'. More precisely, $f_n \asymp g_n$ means $0 < \liminf_n f_n/g_n \le \limsup_n f_n/g_n < \infty$. The last line follows from calculating in polar coordinates. As usual, constants c, C_d, etc. are immaterial to our discussion. In fact, a little more care can be used to show that

$$\mathbb{E}\{J_{n,n}\} \sim C_d \times \begin{cases} 1, & \text{if } d > 4 \\ \ln n, & \text{if } d = 4 \\ n^{\frac{1}{2}}, & \text{if } d = 3 \\ n, & \text{if } d = 2 \\ n^{\frac{3}{2}}, & \text{if } d = 1 \end{cases}. \tag{2.2}$$

We now make the following claim:

Lemma 2.1 *In all dimensions, there exists C_d such that for all $n \ge 1$,*

$$\mathbb{E}\{J_{n,n}^2\} \le C_d \big|\mathbb{E}\{J_{n,n}\}\big|^2.$$

Admitting Lemma 2.1 for the moment, we combine it with (2.2) and the Paley-Zygmund Lemma (Lemma 1.2, Lecture 1) to get:

$$\inf_{n\ge1} \mathbb{P}\big\{J_{n,n} \ge \tfrac{1}{2}\mathbb{E}(J_{n,n})\big\} > 0, \qquad \forall d \ge 1.$$

On the other hand, Eq. (2.2) also shows us that $\mathbb{E}\{J_{n,n}\} \to \infty$ iff $n \le 4$. Thus, we have obtained half of the following theorem, due to Dvoretzky, Erdős and Kakutani:

Theorem 2.2 *Two independent simple walk paths in $\mathbb{Z}^d$ intersect infinitely often iff $d \le 4$.*

As I mentioned earlier, modulo proving Lemma 2.1, we have shown half of this theorem; namely, that $d \le 4$ implies infinite intersections "with positive probability", which implies "with probability one", by the Hewitt–Savage 0-1 law. On the other hand, if $d > 5$, Eq. (2.1) shows us that $\limsup_n \mathbb{E}\{J_{n,n}\} < +\infty$. From this, we gather that $J_{\infty,\infty} < +\infty$, a.s. and we are done. Thus, Theorem 2.2 will follow from our

Proof of Lemma 2.1 Just expand the square, and use symmetry considerations to obtain

$$\begin{aligned}
\mathbb{E}\{J_{n,n}^2\} &= \sum_{i=1}^n \sum_{j=1}^n \sum_{i'=1}^n \sum_{j'=1}^n \mathbb{P}\{X_i^1 = X_j^2\,,\, X_{i'}^1 = X_{j'}^2\} \\
&\le 2 \sum_{1\le i\le i'\le n} \sum_{1\le j\le j'\le n} \mathbb{P}\{X_i^1 = X_j^2\,,\, X_{i'}^1 = X_{j'}^2\}+ \\
&\quad + 2 \sum_{1\le i\le i'\le n} \sum_{1\le j'\le j\le n} \mathbb{P}\{X_i^1 = X_j^2\,,\, X_{i'}^1 = X_{j'}^2\}.
\end{aligned}$$

In expanding the above, it helps to recognize that there are four cases to consider:

1. $i < i'$ and $j < j'$;
2. $i < i'$ but $j > j'$;
3. $i = i'$ but $j < j'$; and finally
4. $i = i'$ and $j = j'$.

Thus, we can write

$$\mathbb{E}\{J_{n,n}^2\} = 2(T_1 + T_2 + T_3 + T_4)$$

where

$$\begin{aligned}
T_1 &= \sum_{1\le i<i'\le n} \sum_{1\le j<j'\le n} \mathbb{P}\{X_i^1 = X_j^2\,,\, X_{i'}^1 = X_{j'}^2\} \\
T_2 &= \sum_{1\le i<i'\le n} \sum_{1\le j'<j\le n} \mathbb{P}\{X_i^1 = X_j^2\,,\, X_{i'}^1 = X_{j'}^2\} \\
T_3 &= \sum_{1\le i\le n} \sum_{1\le j<j'\le n} \mathbb{P}\{X_i^1 = X_j^2 = X_{j'}^2\} \\
T_4 &= \sum_{1\le i\le n} \sum_{1\le j\le n} \mathbb{P}\{X_i^1 = X_j^2\}.
\end{aligned}$$

Of course,

$$T_4 = \mathbb{E}\{J_{n,n}\}. \tag{2.3}$$

Furthermore,

$$
\begin{aligned}
T_3 &= \sum_{1\le i\le n}\sum_{1\le j<j'\le n} \mathbb{P}\{X_i^1 = X_j^2\}\cdot\mathbb{P}\{X_{j'-j}^2 = 0\} \\
&\le \mathbb{E}\{J_{n,n}\}\cdot\sum_{j=1}^n \mathbb{P}\{X_j^2 = 0\}.
\end{aligned}
\tag{2.4}
$$

For T_2, we note that if $i < i'$ and $j' < j$,

$$
\mathbb{P}\{X_i^1 = X_j^2\,,\, X_{i'}^1 = X_{j'}^2\} = \mathbb{P}\{X_i^1 = X_{j'}^2 + \xi_1\,,\, X_i^1 + \xi_2 = X_{j'}^2\}
$$

where

$$
\begin{aligned}
&\xi_1 = X_j^2 - X_{j'}^2, \text{ and} \\
&\xi_2 = X_{i'}^1 - X_i^1.
\end{aligned}
$$

Thus, by the independence of (ξ_1, ξ_2) from $(X_i^1, X_{j'}^2)$,

$$
\begin{aligned}
\mathbb{P}\{X_i^1 = X_j^2\,,\, X_{i'}^1 = X_{j'}^2\} &= \mathbb{P}\{X_i^1 = X_{j'}^2 + \xi_1\,,\, \xi_1 + \xi_2 = 0\} \\
&\le \sup_{a\in\mathbb{Z}^d} \mathbb{P}\{X_i^1 = X_{j'}^2 + a\}\cdot\mathbb{P}\{\xi_1 + \xi_2 = 0\} \\
&= \sup_{a\in\mathbb{Z}^d} \mathbb{P}\{X_i^1 = X_{j'}^2 + a\}\cdot\mathbb{P}\{X_{i'-i}^1 + X_{j-j'}^2 = 0\} \\
&= \sup_{a\in\mathbb{Z}^d} \mathbb{P}\{X_i^1 = X_{j'}^2 + a\}\cdot\mathbb{P}\{X_{i'-i}^1 = X_{j-j'}^2\},
\end{aligned}
$$

Now, by another appeal to the local limit theorem, $\mathbb{P}\{X_i^1 = X_j^2 + a\} \le C_d(i+j)^{-\frac{d}{2}} \le C_d'\mathbb{P}\{X_i^1 = X_j^2\}$, at least when $i+j$ is even; cf. (2.1). Therefore, by reshuffling the labels,

$$
\begin{aligned}
T_2 &\le \sum\sum_{1\le i<i'\le n}\sum\sum_{1\le j'<j\le n} \mathbb{P}\{X_i^1 = X_j^2\,,\, X_{i'}^1 = X_{j'}^2\} \\
&\le C_d \sum\sum_{1\le i<i'\le n}\sum\sum_{1\le j'<j\le n} \mathbb{P}\{X_i^1 = X_j^2\}\cdot\mathbb{P}\{X_{i'-i}^1 = X_{j-j'}^2\} \\
&\le C_d \big|\mathbb{E}\{J_{n,n}\}\big|^2.
\end{aligned}
\tag{2.5}
$$

Finally,

$$
\begin{aligned}
T_1 &= \sum_{1\le i<i'\le n}\sum_{1\le j<j'\le n} \mathbb{P}\{X_i^1 = X_j^2\}\mathbb{P}\{X_{i'-i}^1 = X_{j'-j}^2\} \\
&\le \tfrac{1}{2}\big[\mathbb{E}\{J_{n,n}\}\big]^2.
\end{aligned}
$$

Combining this with Eq.'s (2.3), (2.4) and (2.5), we obtain

$$\mathbb{E}\{J_{n,n}^2\} \le C_d\big[\mathbb{E}\{J_{n,n}\}\big]^2 + 2\mathbb{E}\{J_{n,n}\} \cdot \Big[1 + 2\sum_{j=1}^n \mathbb{P}\{X_i^1 = 0\}\Big].$$

We use the local limit theorem, once more, to see that

$$\sum_{j=1}^n \mathbb{P}\{X_j^1 = 0\} \le C_d \times \begin{cases} 1, & \text{if } d \ge 3 \\ \ln n, & \text{if } d = 2 \\ n^{\frac{1}{2}}, & \text{if } d = 1 \end{cases}. \tag{2.6}$$

The lemma follows from this. □

You should compare Theorem 2.2 with the recurrence theorem for Markov chains, specialized to this setting; cf. Theorem 1.1 of Lecture 1. Indeed, we note that by the latter theorem, and by the computations that lead to Eq. (2.6),

Theorem 2.3 *The simple walk in $\mathbb{Z}^d$ is recurrent iff $d \le 2$.*

Thus, it is the case that some transient walks intersect infinitely many times.

We could have studied this earlier on, since there are no multiparameter martingales needed. However, the continuous analogue of this theorem forces us to reconsider such a remark. Indeed, we shall prove the following theorem of Dvoretzky et al. next time, using the Cairoli-Walsh theory.

Theorem 2.4 *Let X and Y denote 2 independent d-dimensional Brownian motions, both starting at 0. Then, $X(\mathbb{R}_+) \cap Y(\mathbb{R}_+) = \{0\}$ iff $d \ge 4$.*

In the course of our proof of Lévy theorem, we needed a probability estimate (cf. Theorem 1.2 of Lecture 2) that has the following two-parameter analogue, due to Aizenmann and Simon.

Theorem 2.5 (Aizenmann and Simon) *There exists C_1 and C_2, such that for all $\varepsilon \in (0,1)$,*

$$C_1\kappa(\varepsilon) \le \mathbb{P}\{\inf_{1\le s,t\le 2} |X(s) - Y(t)| \le \varepsilon\} \le C_2\kappa(\varepsilon),$$

where

$$\kappa(\varepsilon) = \begin{cases} \varepsilon^{d-4}, & \textit{if } d \ge 5 \\ [\ln(1/\varepsilon)]^{-1}, & \textit{if } d = 4 \\ 1, & \textit{if } d \le 3 \end{cases}.$$

Lecture 6

Intersections of Brownian Motion, Multiparameter Martingales

Before proceeding, I wish to state two open problems; one of them was mentioned in Lecture 3.

Open Problem 1. Suppose $X_{i,j}$ are i.i.d. centered random variables for $i, j \geq 1$, and consider the 2-parameter random walk

$$S_{n,m} = \sum_{i \leq n} \sum_{j \leq m} X_{i,j}, \quad \forall n, m \geq 1.$$

By standardizing them, we can assume, without loss of generality, that $\mathbb{E}[X_{i,j}] = 0$ and $\mathbb{E}[X_{i,j}^2] = 1$. Then, it is possible to show that the following law of the iterated logarithm holds:

$$\limsup_{n,m \to \infty} \frac{S_{n,m}}{\sqrt{4nm \log \log nm}} = 1,$$

at least as long as $X_{1,1} \in L^2 \ln L$; cf. Ch. 4 of MPP. On the other hand, $X_{1,1} \in L^2$ is clearly not enough. In fact, $X_{1,1} \in \frac{L^2 \ln L}{\ln \ln L}$ is necessary for

$$\limsup_{n,m \to \infty} \frac{|S_{n,m}|}{\sqrt{nm \log \log nm}} < +\infty,$$

(*loc. cit.*) Is there a necessary and sufficient moment-type condition for the LIL?

Open Problem 2. By working harder, one can extend Theorem 2.2 of Lecture 5 as follows: if S^1 and S^2 are two independent, *symmetric*, and transient random walks on $\mathbb{Z}^d$,

$$\#\{S^1(\mathbb{Z}_+) \cap S^2(\mathbb{Z}_+)\} = +\infty \iff \sum_{i,j\geq 1} \mathbb{P}\{S^1_i = S^2_j\} = +\infty.$$

See Ch. 3 of MPP. Can symmetry be dropped? Are there *any* nonsymmetric walks that can be analyzed?

1 Proof of Theorem 2.2

We start, as before, by considering the first two moments of

$$J_\varepsilon = \int_1^2 \int_1^2 \mathbf{1}_{\{|X_s - Y_t| \leq \varepsilon\}}\, ds\, dt.$$

Note that $X_s - Y_t$ is a normal vector with mean zero and variance matrix $\sqrt{s+t}$ times the identity matrix. Note also that $1 \leq \sqrt{s+t} \leq 2$, for all $s, t \in [1,2]$. Thus, by Lemma 1.1 of Lecture 3, we can find C_1 and C_2 such that for all $\varepsilon \in (0,1)$,

$$C_1 \varepsilon^d \leq \mathbb{E}\{J_\varepsilon\} \leq C_2 \varepsilon^d.$$

Furthermore,

$$\begin{aligned}
\mathbb{E}\{J_\varepsilon^2\} &= \iint_{[1,2]^2} \iint_{[1,2]^2} \mathbb{P}\{|X_s - Y_t| \leq \varepsilon\,,\ |X_{s'} - Y_{t'}| \leq \varepsilon\}\, ds\, dt\, ds'\, dt' \\
&= 2(T_1 + T_2),
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \iint_{1 \leq s \leq s' \leq 2} \iint_{1 \leq t \leq t' \leq 2} \mathbb{P}\{|X_s - Y_t| \leq \varepsilon\,,\ |X_{s'} - Y_{t'}| \leq \varepsilon\}\, ds\, dt\, ds'\, dt', \text{ and} \\
T_2 &= \iint_{1 \leq s' \leq s \leq 2} \iint_{1 \leq t \leq t' \leq 2} \mathbb{P}\{|X_s - Y_t| \leq \varepsilon\,,\ |X_{s'} - Y_{t'}| \leq \varepsilon\}\, ds\, dt\, ds'\, dt'.
\end{aligned}$$

Clearly,

$$\begin{aligned}
T_1 &\leq \iint_{1 \leq s \leq s' \leq 2} \iint_{1 \leq t \leq t' \leq 2} \mathbb{P}\{|X_s - Y_t| \leq \varepsilon\} \cdot \mathbb{P}\{|X_{s'-s} - Y_{t'-t}| \leq 2\varepsilon\}\, ds\, dt\, ds'\, dt' \\
&\leq C_3 \iint_{1 \leq s \leq s' \leq 2} \iint_{1 \leq t \leq t' \leq 2} \varepsilon^d \cdot \left(\frac{\varepsilon}{\sqrt{|s'-s| + |t'-t|}} \wedge 1 \right)^d ds\, dt\, ds'\, dt'.
\end{aligned}$$

We have used Lemma 1.1 of Lecture 3 once more. Now, this shows that

$$\begin{aligned}
T_1 &\leq C_3' \varepsilon^d \int_{\substack{x\in\mathbb{R}^2:\\ \|x\|\leq c}} \Big(\frac{\varepsilon}{\|x\|^{\frac{1}{2}} \wedge 1}\Big)^d dx \\
&= C_3'' \varepsilon^d \int_0^c \Big(\frac{\varepsilon}{r^{\frac{1}{2}}} \wedge 1\Big)^d r\, dr \\
&\leq C_3^\circ \varepsilon^d \times \begin{cases} \varepsilon^d, & \text{if } d \leq 3 \\ \varepsilon^2 \ln_+(\dfrac{1}{\varepsilon}), & \text{if } d = 4 \\ \varepsilon^2, & \text{if } d \geq 5 \end{cases}.
\end{aligned}$$

We will obtain the same estimates for T_2:

$$\begin{aligned}
T_2 &= \iint\limits_{1\leq s'\leq s\leq 2} \iint\limits_{1\leq t\leq t'\leq 2} \mathbb{P}\{|X_{s'} + \xi_1 - Y_t| \leq \varepsilon\,,\, |X_{s'} + \xi_2 - Y_t| \leq \varepsilon\}\, ds\, dt\, ds'\, dt' \\
&\leq \iint\limits_{1\leq s'\leq s\leq 2} \iint\limits_{1\leq t\leq t'\leq 2} \mathbb{P}\{|X_{s'} + \xi_1 - Y_t| \leq \varepsilon\,,\, |\xi_1 + \xi_2| \leq 2\varepsilon\}\, ds\, dt\, ds'\, dt',
\end{aligned}$$

where

$$\xi_1 = (X_s - X_{s'}) \qquad \text{and } \xi_2 = (Y_{t'} - Y_t).$$

Note that $X_s - Y_t$ is independent of (ξ_1, ξ_2). Thus,

$$\begin{aligned}
T_2 &\leq \iint\limits_{1\leq s'\leq s\leq 2} \iint\limits_{1\leq t\leq t'\leq 2} \sup_{a\in\mathbb{R}^d} \mathbb{P}\{|X_{s'} + a - Y_t| \leq \varepsilon\} \mathbb{P}\{|\xi_1 + \xi_2| \leq 2\varepsilon\}\, ds\, dt\, ds'\, dt' \\
&\leq C_4 \varepsilon^d \times \begin{cases} \varepsilon^d, & \text{if } d \leq 3 \\ \varepsilon^2 \ln_+(\dfrac{1}{\varepsilon}), & \text{if } d = 4 \\ \varepsilon^2, & \text{if } d \geq 5 \end{cases},
\end{aligned}$$

by yet another application of Lemma 1.1 of Lecture 3. Combining what we have, we obtain

$$\mathbb{E}\{J_\varepsilon\} \asymp \varepsilon^d \qquad \text{and} \qquad \mathbb{E}\{J_\varepsilon^2\} \leq C\varepsilon^d \times \begin{cases} \varepsilon^d, & \text{if } d \leq 3 \\ \varepsilon^4 \ln_+(\dfrac{1}{\varepsilon}), & \text{if } d = 4 \\ \varepsilon^4, & \text{if } d \geq 5 \end{cases}. \tag{1.1}$$

This is good enough to prove that

$$\mathbb{P}\{\inf_{1\leq s,t\leq 2} |X_s - Y_t| \leq \varepsilon\} \geq C\kappa(\varepsilon),$$

which is the desired lower bound. We now show this is sharp.

Consider the two 1-parameter filtrations

$$\mathcal{F}_s^1 = \sigma(X_r;\, r \le s) \vee \sigma(Y_u;\, u \ge 0) \qquad \text{and} \qquad \mathcal{F}_t^2 = \sigma(Y_r;\, r \le t) \vee \sigma(X_u;\, u \ge 0).$$

We now estimate

$$\begin{aligned} M_{s,t} = \mathbb{E}\{J_\varepsilon \,|\, \mathcal{F}_s^1 \cap \mathcal{F}_t^2\} &\ge \int_s^2 \int_t^2 \mathbb{P}\{|X_u - Y_v| \le \varepsilon \,|\, \mathcal{F}_s^1 \cap \mathcal{F}_t^2\}\, dv\, du \cdot \mathbf{1}_{\{|X_s - Y_t| \le \frac{\varepsilon}{2}\}} \\ &\ge \int_0^{2-s} \int_0^{2-t} \mathbb{P}\{|X_{u-s} - Y_{v-t}| \le \tfrac{\varepsilon}{2}\}\, dv\, du \cdot \mathbf{1}_{\{|X_s - Y_t| \le \frac{\varepsilon}{2}\}}. \end{aligned}$$

Thus, as long as $1 \le s, t \le \frac{3}{2}$,

$$M_{s,t} \ge \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \mathbb{P}\{|X_{u-s} - Y_{v-t}| \le \tfrac{\varepsilon}{2}\}\, dv\, du \cdot \mathbf{1}_{\{|X_s - Y_t| \le \frac{\varepsilon}{2}\}}.$$

Now, the above double integral is bounded below by a constant multiple of

$$\int_{\substack{z \in \mathbb{R}^2: \\ \|z\| \le c}} \Big(\frac{\varepsilon}{\|z\|^{\frac{1}{2}}} \wedge 1 \Big)^d dz \ge C_d \frac{\mathbb{E}\{J_\varepsilon^2\}}{\varepsilon^d}.$$

So,

$$\mathbb{E}\{\sup_{1 \le s,t \le \frac{3}{2}} M_{s,t}^2\} \ge C_d \Big[\frac{\mathbb{E}\{J_\varepsilon^2\}}{\varepsilon^d}\Big]^2 \mathbb{P}\{\inf_{1 \le s,t \le \frac{3}{2}} |X_s - Y_t| \le \tfrac{\varepsilon}{2}\}.$$

On the other hand, Cairoli's maximal inequality asssures us that $\mathbb{E}\{\sup_{s,t} M_{s,t}^2\} \le 16\mathbb{E}\{J_\varepsilon^2\}$, which, together with Eq. (1.1) yields

$$\mathbb{P}\{\inf_{1 \le s,t \le \frac{3}{2}} |X_s - Y_t| \le \tfrac{\varepsilon}{2}\} \le C\kappa(\varepsilon).$$

By a change of scale, the infimum can be taken over $s, t \in [1, 2]$ at little cost, and we are done. $\square$

2 Multiparameter Martingales

Consider an N-parameter process $\{M_t;\, t \in \mathbb{N}^N\}$, and an N-parameter sequence of sigma-fields $\mathcal{F} = \{\mathcal{F}_t;\, t \in \mathbb{N}^N\}$.

We say that $\mathcal{F}$ is a *filtration*, if

$$s \preccurlyeq t \implies \mathcal{F}_s \subseteq \mathcal{F}_t.$$

Perhaps it is best to recall that $s \preccurlyeq t$ if and only if $s_i \le t_i$ for all $i = 1, \ldots, N$. We say that M is *adapted* to the N-parameter filtration $\mathcal{F}$ if for all $t \in \mathbb{N}^N$, M_t is $\mathcal{F}_t$-measurable. Finally, M is a *martingale* with respect to the filtration $\mathcal{F}$ if (i) M is adapted to $\mathcal{F}$; (ii) $M_t \in L^1(\mathbb{P})$ for all $t \in \mathbb{N}^N$; and (iii) almost surely,

$$s \preccurlyeq t \implies \mathbb{E}\{M_t \,|\, \mathcal{F}_s\} = M_s.$$

In general, multiparameter martingales have not much structure to speak of. However, they do, if the filtration in question has some conditional independence properties.

2.1 Commutation

We say that the N-parameter filtration $\mathcal{F}$ is *commuting*, if for all $s, t \in \mathbb{N}^N$, and for all $\mathcal{F}_t$-measurable bounded random variables Y,

$$\mathbb{E}\{Y|\mathcal{F}_s\} = \mathbb{E}\{Y|\mathcal{F}_{s \curlywedge t}\},$$

where $s \curlywedge t$ is the point in $\mathbb{N}^N$ whose ith coordinate is $s_i \wedge t_i$. The typical N-parameter filtration does *not* commute!

Theorem 2.1 *If $\mathcal{F}$ is a commuting N-parameter filtration, for all random variables $Z \in L^1(\mathbb{P})$, and for all $t \in \mathbb{N}^N$,*

$$\mathbb{E}\{Z|\mathcal{F}_t\} = \mathbb{E}\Big[\ \cdots \mathbb{E}\big(\ \mathbb{E}\{Z|\mathcal{F}^1_{t_1}\}\ \big|\ \mathcal{F}^2_{t_2}\big)\ \cdots \Big|\ \mathcal{F}^N_{t_N}\Big].$$

In particular, we shall see, as a consequence, that any martingale with respect to a commuting $\mathcal{F}$ is an ortho-martingale! This, in turn, implies maximal inequalities, convergence theorems, etc.

Proof It suffices to prove this for bounded Z (why?). We will do this when $N = 2$.

By Doob's 1-parameter martingale convergence theorem, and by Lebesgue's dominated convergence theorem,

$$\begin{aligned}
\mathbb{E}\big[\ \mathbb{E}\{Z \,|\, \mathcal{F}^1_i\}\,\big|\,\mathcal{F}^2_j\big] &= \mathbb{E}\big[\lim_{k\to\infty} \mathbb{E}\{Z \,|\, \mathcal{F}_{i,k}\}\,\big|\,\mathcal{F}^2_j\big] \\
&= \lim_{k\to\infty} \mathbb{E}\big[\ \mathbb{E}\{Z \,|\, \mathcal{F}_{i,k}\}\,\big|\,\mathcal{F}^2_j\big] \\
&= \lim_{k\to\infty} \lim_{\ell\to\infty} \mathbb{E}\big[\ \mathbb{E}\{Z \,|\, \mathcal{F}_{i,k}\}\,\big|\,\mathcal{F}_{\ell,j}\big],
\end{aligned}$$

where all of the convergences are taking place in $L^1(\mathbb{P})$. Recall that $\mathcal{F}$ is commuting and $Y = \mathbb{E}[Z \,|\, \mathcal{F}_{i,k}]$ is $\mathcal{F}_{i,k}$-measurable and bounded. This implies that for any other $\ell, j \ge 1$, a.s.,

$$\begin{aligned}
\mathbb{E}[Y \,|\, \mathcal{F}_{\ell,j}] &= \mathbb{E}[Y \,|\, \mathcal{F}_{(i,k)\curlywedge(\ell,j)}] \\
&= \mathbb{E}[Y \,|\, \mathcal{F}_{i\wedge\ell, k\wedge j}] \\
&= \mathbb{E}[Z \,|\, \mathcal{F}_{i\wedge\ell, k\wedge j}].
\end{aligned}$$

Thus, for every $i, j \geq 0$ and for all bounded random variables Z,

$$\mathbb{E}\big[\, \mathbb{E}\{Z \mid \mathcal{F}_i^1\} \,\big|\, \mathcal{F}_j^2\big] = \lim_{k\to\infty} \lim_{\ell\to\infty} \mathbb{E}[Z \mid \mathcal{F}_{i\wedge\ell, k\wedge j}], \qquad \text{a.s.}$$

This is clearly equal to $\mathbb{E}\{Z|\mathcal{F}_{i,j}\}$, and the result follows. $\square$

The notion of commutation is equivalent to conditional independence. Recall that two sigma-fields $\mathcal{G}_1$ and $\mathcal{G}_2$ are *conditionally independent, given* $\mathcal{G}$, if for all bounded $\mathcal{G}_i$-measurable random variables Y_i ($i = 1, 2$),

$$\mathbb{E}\{Y_1 Y_2|\mathcal{G}\} = \mathbb{E}\{Y_1|\mathcal{G}\} \cdot \mathbb{E}\{Y_2|\mathcal{G}\}.$$

Theorem 2.2 *For a given filtration $\mathcal{F} = (\mathcal{F}_t;\ t \in \mathbb{N}_0^N)$, the following are equivalent:*

(i) $\mathcal{F}$ is commuting; and

(ii) for all $s, t \in \mathbb{N}_0^N$, $\mathcal{F}_t$ and $\mathcal{F}_s$ are conditionally independent, given $\mathcal{F}_{s \curlywedge t}$.

Proof Suppose for all $t \in \mathbb{N}_0^N$, Y_t is a bounded $\mathcal{F}_t$-measurable random variable. Then,

$$\mathbb{E}[Y_t Y_s \mid \mathcal{F}_{t\curlywedge s}] = \mathbb{E}\big[Y_t\, \mathbb{E}\{Y_s \mid \mathcal{F}_t\} \,\big|\, \mathcal{F}_{t\curlywedge s}\big] = \mathbb{E}\big[Y_t\, \mathbb{E}\{Y_s \mid \mathcal{F}_{t\curlywedge s}\} \,\big|\, \mathcal{F}_{t\curlywedge s}\big], \qquad \text{a.s.}$$

Thus, $(i) \Rightarrow (ii)$. Conversely, supposing that (ii) holds,

$$\begin{aligned} \mathbb{E}[Y_t Y_s] &= \mathbb{E}\Big[\mathbb{E}\{Y_t Y_s \mid \mathcal{F}_{t\curlywedge s}\}\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\{Y_t \mid \mathcal{F}_{t\curlywedge s}\} \cdot \mathbb{E}\{Y_s \mid \mathcal{F}_{t\curlywedge s}\}\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\{Y_t \mid \mathcal{F}_{t\curlywedge s}\} \cdot Y_s\Big]. \end{aligned}$$

Since $\mathcal{F}_{t\curlywedge s} \subset \mathcal{F}_s$ and the above holds for all bounded, $\mathcal{F}_s$-measurable random variables Y_s, $\mathbb{E}[Y_t \mid \mathcal{F}_{t\curlywedge s}] = \mathbb{E}[Y_t \mid \mathcal{F}_s]$, a.s.. This shows that (ii) implies (i) and hence, (ii) is equivalent to (i). $\square$

Example If $X_{i,j}$ are independent random variables ($i, j \geq 1$), define $\mathcal{F}_{n,m} = \sigma(X_{i,j};\ i \leq n, j \leq m)$ to see that $\mathcal{F}$ is commuting. In particular, if we also knew that the X's have mean zero, $S_{n,m} = \sum_{i\leq n} \sum_{j\leq m} X_{i,j}$ is a martingale, as well as an ortho-martingale. What about additive walks? $\square$

2.2 Back to Martingales

We conclude our discussion of martingales, under commutation, by linking them to ortho-martingales.

Theorem 2.3 *Suppose $\mathcal{F}$ is a commuting N-parameter filtration. Then, M is a martingale with respect to $\mathcal{F}$ if and only if it is an ortho-martingale with respect to the marginal filtrations of $\mathcal{F}$.*

Proof Once again, we only really need to do this for $N = 2$. First, suppose M is an ortho-martingale, and consider $i \leq n$ and $j \leq m$. Clearly,

$$\mathbb{E}\{M_{n,m} | \mathcal{F}_{i,j}\} = \mathbb{E}\big(\mathbb{E}\{M_{n,m} | \mathcal{F}_i^1\} \bigm| \mathcal{F}_{i,j}\big) = \mathbb{E}\{M_{i,m} | \mathcal{F}_{i,j}\}.$$

Now, project m to j to see that orthomartingale implies martingale. In fact, for this half, commutation is not needed.

Conversely, suppose M is a martingale with respect to a commuting filtration. We will show it is an ortho-martingale for the marginal filtrations. By Doob's convergence theorem,

$$\mathbb{E}\{M_{i+1,j} | \mathcal{F}_i^1\} = \lim_{k \to \infty} \mathbb{E}\{M_{i+1,j} | \mathcal{F}_{i,k}\}.$$

Thanks to commutation, the above is $\lim_k \mathbb{E}\{M_{i+1,j} | \mathcal{F}_{i,j}\} = M_{i,j}$. Similarly, $\mathbb{E}\{M_{i,j+1} | \mathcal{F}_j^2\} = M_{i,j}$, and we are done. $\square$

Thus, in the presence of commutation, we have maximal inequalities, convergence theorems, etc.

Lecture 7

Capacity, Energy and Dimension

We now come to the second part of these lectures which has to do with "exceptional sets". The most obvious class of exceptional sets are those of measure 0, where the measure is some nice one. As an example, consider a compact set $E \subset \mathbb{R}^d$. One way to construct its Lebesgue measure is as follows: cover E by small boxes, compute the volume of the cover, and then optimize over all the covers. That is,

$$|E| = \lim_{\varepsilon \to 0^+} \inf \left\{ \sum_i [\mathrm{diam}(E_i)]^d : \ E_1, E_2, \ldots \text{ closed boxes of diameter} \le \varepsilon \text{ with } \cup_i E_i \supseteq E \right\}.$$

Here, we are computing the diameter of the box as twice its ℓ^1-radius; i.e., it is the length of any side. This is equivalent to the usual definition of Lebesgue's measure, although it is long out of fashion in standard analysis courses.

1 Hausdorff Dimension and Measures

The first class of exceptional sets that we can discuss are those of Lebesgue's measure 0, of course. But, this is too crude for differentiating amongst very thin sets. For example, consider the rationals $\mathbb{Q}$, as well as Cantor's tertiary set $\mathbf{C}$. While they are both measure 0 sets, $\mathbf{C}$ is uncountable, whereas $\mathbb{Q}$ is not. We would like a concrete way of saying that $\mathbf{C}$ is larger than $\mathbb{Q}$, and perhaps measure how much larger, as well. There are many ways of doing this, and we will choose a route that is useful for our probabilistic needs. First, note that for any $\alpha \ge 0$, we can define the analogue of $|E|$ as above. Namely, define

$$\mathcal{H}_\alpha(E) = \lim_{\varepsilon \to 0^+} \inf \left\{ \sum_i [\mathrm{diam}(E_i)]^\alpha : \ E_1, E_2, \ldots \text{ closed boxes of diameter} \le \varepsilon \text{ with } \cup_i E_i \supseteq E \right\}.$$

This makes sense even if $\alpha \le 0$.

The set function $\mathcal{H}_\alpha$ is called the α-dimensional *Hausdorff measure*. This terminology is motivated by the following, which is proved by using the method given to us by Carathéodory:

Theorem 1.1 *The set function $\mathcal{H}_\alpha$ is an outer measure on Borel subsets of $\mathbb{R}^d$. For all $\alpha > d$, $\mathcal{H}_\alpha(E) = 0$ identically. On the other hand, when $\alpha \le d$ is an integer, $\mathcal{H}_\alpha(E)$ equals the α-dimensional Lebesgue's measure of Borel set E.*

Hausdorff dimensions provide us with a more refined sense of how big a set is. Note that for any compact (or even Borel, say) set E, there is *always* a critical α such that for all $\beta < \alpha$, $\mathcal{H}_\beta(E) = 0$, while for all $\beta > \alpha$, $\mathcal{H}_\beta(E) = +\infty$. This is an easy calculation. But it leads to the following important measure-theoretic notion of dimension:

$$\dim(E) = \inf\{\alpha : \mathcal{H}_\alpha(E) = 0\} = \sup\{\alpha : \mathcal{H}_\alpha(E) = +\infty\}.$$

This is the *Hausdorff dimension* of E.

How does one compute the Hausdorff dimension of a set? You typically proceed by establishing an upper bound, as well as a lower bound. The first step is not hard: just find a "good" covering E_i of diameter less than ε, and compute $\sum_i [\text{diam}(E_i)]^\alpha$. Here is one way to get an upper bound systematically; other ways abound.

Suppose we are interested in computing the Hausdorff dimension of a given compact set $E \subset [0,1]^d$. Fix a *real* number $n \ge 1$, and define $E_j = [\frac{j}{n}, \frac{j+1}{n}[$, for integers $0 \le j \le n$. Then, it is clear that the diameter of each E_j is no more than $\frac{2}{n}$, while $\cup_j E_j \supset E$. So,

$$\mathcal{H}_\alpha(E) \le (\frac{2}{n})^\alpha \mathcal{N}_n(E),$$

where $\mathcal{N}_n(E) = \sum_{0 \le j \le n} \mathbf{1}\{I_{j,n} \cap E \ne \varnothing\}$ is the number of times the intervals $I_{j,n}$ contains portions of E. Therefore, if we can find α such that $\limsup_n n^{-\alpha} \mathcal{N}_n(E) < +\infty$, we have $\dim(E) \le \alpha$.[1] Incidentally, the minimal α such that $\limsup_n n^{-\alpha}\mathcal{N}_n(E) < +\infty$ is the so-called *upper Minkowski (or box) dimension* of E. If we write the latter as $\dim_{_M}(E)$, we have shown that

$$\dim(E) \le \dim_{_M}(E). \tag{1.1}$$

If we replace E_j by a d-dimensional box of the form $[\frac{i_1}{n}, \frac{i_1+1}{n}[\times \cdots \times [\frac{i_d}{n}, \frac{i_d+1}{n}[$ and repeat the procedure, we obtain the upper Minkowski dimension in d dimensions, and Eq. (1.1) remains to hold.

We now use this to obtain an upper bound for the tertiary Cantor set $\mathbf{C}$. First, let us recall the following iterative construction of $\mathbf{C}$: let $\mathbf{C}_0 = [0,1]$. Now, remove the middle third to obtain $\mathbf{C}_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Next, remove the middle thirds of each of the two subintervals to get $\mathbf{C}_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and so on. In this way, you have a decreasing sequence of compact subsets of $[0,1]$, and, as such, $\mathbf{C} = \cap_n \mathbf{C}_n$ is a nontrivial compact subset of $[0,1]$. At the nth level of construction, $\mathbf{C}_n$ is comprised of 2^n intervals of length

[1]We do not require n to be an integer here.

3^{-n}. Therefore, $|\mathbf{C}_n| = (\frac{2}{3})^n \to |\mathbf{C}| = 0$. On the other hand, we just argued that there are 2^n boxes, of diameter no greater than (in fact, equal to) 3^n that cover $\mathbf{C}$. Therefore, we have shown that $\mathcal{N}_{3^n}(\mathbf{C}) = 2^n$. In particular, for any $\alpha > \log_3(2)$, $\limsup_{m\to\infty}(3^{-m})^{-\alpha}\mathcal{N}_{3^m}(E) = \lim_{m\to\infty} 3^{-m\alpha}2^m = 0$. So that, after a little work, we get $\dim_{_M}(E) \le \log_3(2)$. In fact, it is easy to see, by the same reasoning, that $\dim_M(E) = \log_3(2)$. In any event, we obtain the following:

$$\dim(\mathbf{C}) \le \log_3(2) = \frac{\ln 2}{\ln 3}. \tag{1.2}$$

We will show that this is sharp in that the above inequality is an equality. But first, a question: *why not stick to Minkowski dimension?* It is certainly easier to compute than Hausdorff dimension, and at first sight, more natural. To answer this, try computing $\dim_{_M}(\mathbb{Q})$, or $\dim_{_M}$ of any other dense subset of $[0,1]^d$ for that matter! You will see that the answer is 1! On the other hand, it is not hard to show that $\dim(E) = 0$ if E is countable, for then we can write $E = \{r_i\}$ and note that $\{r_i\}$ is a cover of E with diameter less than ε. This seemingly technical difference is really a big one.

Now, to the lower bound for $\dim(\mathbf{C})$. Obtaining lower bound on Hausdorff dimension is, in principle, very hard, since you have to work uniformly over all covers. What makes things difficult is that there are *alot* of potential covers!

The ingeneous idea behind obtaining lower bounds is due to O. Frostman who found it in his Ph.D. thesis in the 1935! Namely,

Theorem 1.2 (Frostman's lemma) *Suppose we knew that the compact set E carries a probability measure μ that is Hölder-smooth in the following sense: there exists $\alpha > 0$ and a constant C such that for all $r \in (0,1)$,*

$$\mu(\mathcal{B}(y,r)) \le Cr^\alpha,$$

for μ-almost all y, where $\mathcal{B}(y,r)$ is the ℓ^∞-ball of radius r about $y \in \mathbb{R}^d$. Then, $\dim(E) \ge \alpha$.

There is a converse to this that we will only need once, and will not prove, as a result; for a proof, see Appendix C of MPP.

Theorem 1.3 (Frostman's Lemma (continued)) *Suppose $\dim(E) \ge \alpha > 0$. Then, for each $\beta < \alpha$, there exists $\mu \in \mathcal{P}(E)$ such that*

$$\sup_{x\in\mathbb{R}^d}\sup_{r\in(0,1)} \frac{\mu\{\mathcal{B}(x,r)\}}{r^\beta} < +\infty.$$

Proof I will prove this when instead of μ-almost all x, the lemma holds for all x. The necessary modifications to prove the general case are technical but not hard.

Fix $\varepsilon \in (0,1)$, and consider any cover $E_1, E_2, \ldots$ of diameter $\leq \varepsilon$. Note that

$$1 = \mu(E) \leq \sum_i \mu(E_i) \leq C \sum_i \left[\text{diam}(E_j)\right]^\alpha.$$

Optimize over all such covers, and let $\varepsilon \to 0$, to see that $1 \leq 2C\mathcal{H}_\alpha(E)$. The theorem follows, since this shows that for any $\beta < \alpha$, $\mathcal{H}_\beta(E) = +\infty$. (To prove in the general case, note that if $\mu(E_j)$ is not less than $C[\text{diam}(E_j)]^\alpha$, we can cover E_j by at most 2^d compact intervals $F_{j,1}, \ldots, F_{j,2^d}$ of diameter less than twice that of E_j, such that $\mu(F_{j,k}) \leq C[\text{diam}(F_{j,k})]^\alpha \leq 2^\alpha C[\text{diam}(E_j)]^\alpha$. Thus, $\mu(E_j) \leq 2^{\alpha+d}C[\text{diam}(E_j)]^\alpha$, which is good enough.) □

We use this to complete our proof of the following.

Proposition 1.4 *If* $\mathbf{C}$ *denotes the tertiary Cantor set,* $\dim(\mathbf{C}) = \frac{\ln 2}{\ln 3}$.

Proof In light of what we have already done, we only need to verify the lower bound on dimension. We do this by finding a sufficiently smooth measure on $\mathbf{C}$. Our choice is more or less obvious and is found iteratively as follows: construct the smoothest possible probability measure μ_n on $\mathbf{C}_n$ and "take limits". Now, the smoothest and flattest probability measure on $\mathbf{C}_n$ is the uniform measure, μ_n. It is easy to see that for all $x \in [0,1]$,

$$\mu_n\big([x - 3^{-n}, x + 3^{-n}]\big) \leq 2^{-n} = (3^{-n})^{\ln 2/\ln 3}. \tag{1.3}$$

This is suggestive, but we need to work a little bit more. To do so, we next note that the μ_n's are nested: We write $\mathbf{C}_n = \cup_{i=1}^{2^n} I_{i,n}$ where $I_{i,n}$ is an interval of length 3^{-n}. The nested property of the μ_n's is the following, which can be checked by induction:

$$\forall n \geq m, \forall j = 1, \ldots, 2^m: \qquad \mu_n(I_{j,m}) = \mu_m(I_{j,m}) = 2^{-m}.$$

Standard weak convergence theory guarantees us of the existence of a probability measure μ_∞ on the compact set $\mathbf{C}$ such that for all $m \geq 1$ and all $j = 1, \ldots, 2^m$,

$$\mu_\infty(I_{j,m}) = \mu_m(I_{j,m}) = 2^{-m}.$$

Moreover, Eq. (1.3) extends to μ_∞. Namely, for all $x \in [0,1]$ and all $n \geq 0$,

$$\mu_\infty([x - 3^{-n}, x + 3^{-n}]) \leq (3^{-n})^{\ln 2/\ln 3}.$$

Now, if $r \in (0,1)$, we can find $n \geq 0$ such that $3^{-n-1} \leq r \leq 3^{-n}$. Therefore,

$$\sup_x \mu_\infty([x-r, x+r]) \leq \sup_x \mu_\infty([x-3^{-n}, x+3^{-n}]) \leq (3^{-n})^{\ln 2/\ln 3} \leq (3r)^{\ln 2/\ln 3}.$$

So, we have found a probability measure μ_∞ on $\mathbf{C}$, that satisfies the condition of Frostman's lemma with $C = 3^{\ln 2/\ln 3} = 2$ and $\alpha = \ln 3/\ln 3$. This completes our proof. □

2 Energy and Capacity

Suppose μ is a probability measure on some given compact set $E \subset \mathbb{R}^d$. We will write this as $\mu \in \mathcal{P}(E)$, and define for any measurable function $f : E \times E \to \mathbb{R}_+ \cup \{\infty\}$,

$$\mathcal{E}_f(\mu) = \iint f(x,y)\, \mu(dx)\, \mu(dx).$$

This is the *energy* of μ with respect to the gauge function f; it is always defined although it may be infinite. The following energy forms are of use to us:

$$\mathsf{Energy}_\alpha(\mu) = \iint |x-y|^{-\alpha}\, \mu(dx)\, \mu(dy),$$

where $|x| = \max_{1 \leq j \leq d} |x_j|$ for concreteness, although any other Euclidean norm will do just as well. This is the so-called α-dimensional *Bessel–Riesz energy* of μ. The question, in the flavor of the previous section, is *when does a set E carry a probability measure of finite energy?* To facilitate the discussion, we define the *capacity* of a set E by

$$\mathcal{C}_f(E) = \Big[\inf_{\mu \in \mathcal{P}(E)} \mathcal{E}_f(\mu)\Big]^{-1}, \qquad \text{and in particular,}$$

$$\mathsf{Cap}_\alpha(E) = \Big[\inf_{\mu \in \mathcal{P}(E)} \mathsf{Energy}_\alpha(\mu)\Big]^{-1}.$$

The above is Gauss' principle of minimum energy. Next, we argue that there is a minimum energy measure called the equilibrium measure. Moreover, its potential is essentially constant, and the constant is the energy.

Theorem 2.1 (Equilibrium Measure) *Suppose E is a compact set in $\mathbb{R}^d$ such that for some $\alpha > 0$, $\mathsf{Cap}_\alpha(E) > 0$. Then, there exists $\mu \in \mathcal{P}(E)$, such that*

$$\mathsf{Energy}_\alpha(\mu) = \big[\mathsf{Cap}_\alpha(E)\big]^{-1}.$$

Moreover, for μ-almost all x,

$$\int |x-y|^{-\alpha}\, \mu(dy) = \mathsf{Energy}_\alpha(\mu).$$

Proof By definition, there exists a sequence of probability measures μ_n, all supported on E, such that (i) they have finite energy; and (ii) for all $n \geq 1$, $(1 + \frac{1}{n})[\mathsf{Cap}_\alpha(E)]^{-1} \geq \mathsf{Energy}_\alpha(\mu_n) \geq [\mathsf{Cap}_\alpha(E)]^{-1}$. Let μ be any subsequential limit of the μ_n's. Since $\mu \in \mathcal{P}(E)$ as well, $\mathsf{Energy}_\alpha(\mu) \geq [\mathsf{Cap}_\alpha(E)]^{-1}$. We aim to show the converse holds too. By going to a subsequence n' along which $\mu_{n'}$ converges weakly to μ, we see that for any $r_0 > 0$,

$$\iint_{|x-y| \geq r_0} |x-y|^{-\alpha}\, \mu(dx)\, \mu(dy) = \lim_{n' \to \infty} \iint_{|x-y| \geq r_0} |x-y|^{-\alpha}\, \mu_{n'}(dx)\, \mu_{n'}(dy) \leq [\mathsf{Cap}_\alpha(E)]^{-1}.$$

Let $r_0 \downarrow 0$ and use the dominated convergence theorem to deduce the first assertion. For the second assertion, i.e., that the minimum energy principle is actually achieved for some probability measure.

Lecture 8

Frostman's Theorem, Hausdorff Dimension and Brownian Motion

1 Frostman's Theorem (Continued)

Now, consider

$$\Upsilon_\eta = \left\{ x \in E : \int |x-y|^{-\alpha}\, \mu(dy) < (1-\eta)\mathsf{Energy}_\alpha(\mu) \right\}, \qquad \eta \in (0,1).$$

We wish to show that $\mu(\Upsilon_\eta) = 0$ for all $\eta \in (0,1)$. If this is not the case for some $\eta \in (0,1)$, then, consider the following

$$\zeta(\bullet) = \frac{\mu(\bullet \cap \Upsilon_\eta)}{\mu(\Upsilon_\eta)}.$$

Evidently, $\zeta \in \mathcal{P}(E)$, and has finite energy. Define

$$\lambda_\varepsilon = (1-\varepsilon)\mu + \varepsilon\zeta, \qquad \varepsilon \in (0,1).$$

Then, λ_ε is also a probability measure on E, and it, too, has finite energy. In fact, writing $\lambda_\varepsilon = \mu - \varepsilon(\mu - \zeta)$, a little calculation shows that

$$\mathsf{Energy}_\alpha(\lambda_\varepsilon) = \mathsf{Energy}_\alpha(\mu) + \varepsilon^2 \mathsf{Energy}_\alpha(\mu - \zeta) - 2\varepsilon \iint |x-y|^{-\alpha}\, \mu(dx)\, \big[\mu(dy) - \zeta(dy)\big].$$

(The energy of $\mu - \zeta$ is defined as if $\mu - \zeta$ were a positive measure.)

Since μ minimizes energy, the above is greater than or equal to $\mathsf{Energy}_\alpha(\mu)$. Thus,

$$\varepsilon^2 \mathsf{Energy}_\alpha(\mu - \zeta) \geq 2\varepsilon \iint |x-y|^{-\alpha}\, \mu(dx)\, \big[\mu(dy) - \zeta(dy)\big].$$

Divide by ε and let $\varepsilon \to 0$ to see that

$$\mathsf{Energy}_\alpha(\mu) \le \iint |x-y|^{-\alpha}\, \mu(dx)\, \zeta(dy).$$

But by the definition of Υ_η, the right hand side is no more than $(1-\eta)\mathsf{Energy}_\alpha(\mu)$, which contradicts the assumption that $\mu(\Upsilon_\eta) > 0$. In other words,

$$\int |x-y|^{-\alpha}\, \mu(dy) \ge \mathsf{Energy}_\alpha(\mu), \qquad \mu\text{-a.s.}$$

It suffices to show the converse inequality. But this is easy. Indeed, suppose

$$\mathfrak{G}\mu(x) = \int |x-y|^{-\alpha}\, \mu(dy) \ge (1+\eta)\mathsf{Energy}_\alpha(\mu),$$

on a set of positive μ-measure. The function $x \mapsto \mathfrak{G}\mu(x)$ is the *α-dimensional potential* of the measure μ. We could integrate $[d\mu]$ to get the desired contradiction, viz.,

$$\begin{aligned}\mathsf{Energy}_\alpha(\mu) &= \int_{\Theta_\eta} \mathfrak{G}\mu(x)\, \mu(dx) + \int_{\Theta_\eta^{\complement}} \mathfrak{G}\mu(x)\, \mu(dx)\\ &\ge (1+\eta)\mathsf{Energy}_\alpha(\mu)\cdot \mu(\Theta_\eta) + \int_{\Theta_\eta^{\complement}} \mathfrak{G}\mu(x)\, \mu(dx),\end{aligned}$$

where $\Theta_\eta = \{x :\ \mathfrak{G}\mu(x) \ge (1+\eta)\mathsf{Energy}_\alpha(\mu)\}$. Therefore, by Theorem 2.1 on equilibrium measure,

$$\begin{aligned}\mathsf{Energy}_\alpha(\mu) &\ge \mathsf{Energy}_\alpha(\mu)\Big[(1+\eta)\mu(\Theta_\eta) + \mu(\Theta_\eta^{\complement})\Big]\\ &= \mathsf{Energy}_\alpha(\mu)\Big[1 + \eta\mu(\Theta_\eta^{\complement})\Big],\end{aligned}$$

which is a contradiction, unless $\mu(\Theta_\eta) = 0$. This concludes our proof. $\square$

SOMETHING TO TRY: The α-dimensional Bessel–Riesz energy defines a Hilbertian pre-norm. Indeed, define $\mathcal{M}_\alpha(E)$ to be the collection of all measures of finite α-dimensional Bessel–Riesz energy on E. On this, define the inner product,

$$\langle \mu, \nu \rangle = \iint |x-y|^{-\alpha}\, \mu(dx)\, \nu(dy).$$

Check that this defines a positive-definite bilinear form on $\mathcal{M}_\alpha(E)$ if $\alpha \in (0, d)$. From this, conclude that for all $\mu, \nu \in \mathcal{M}_\alpha(E)$, $\langle \mu, \nu \rangle^2 \le \mathsf{Energy}_\alpha(\mu) \cdot \mathsf{Energy}_\alpha(\nu)$. This fills a gap in the above proof.

The *capacitary dimension* of a compact set $E \subset \mathbb{R}^d$ is defined as

$$\dim_c(E) = \sup\big\{\alpha :\ \mathsf{Cap}_\alpha(E) > 0\big\} = \inf\big\{\alpha :\ \mathsf{Cap}_\alpha(E) = 0\big\}.$$

Theorem 1.1 (Frostman's Theorem) *Capacitary and Hausdorff dimensions are one and the same.*

Proof Here is one half of the proof: we will show that if there exists $\alpha > 0$ and a probability measure μ on E, such that $\mathsf{Energy}_\alpha(\mu) < +\infty \ \Rightarrow \ \dim(E) \geq \alpha$. This shows that $\dim_c(E) \leq \dim(E)$, which is half the theorem.

By Theorem 2.1, we can assume without loss of generality that μ is an equilibrium measure. In particular,

$$\mu(\mathcal{B}(x,r)) \leq r^\alpha \int |x-y|^{-\alpha}\, \mu(dy) = r^\alpha \mathsf{Energy}_\alpha(\mu),$$

μ-almost everywhere. Frostman's lemma (Theorem 1.2) shows that $\dim(E) \geq \alpha$, as needed.

For the other half, we envoke the second half of Frostman's theorem (Theorem 1.3) to produce for each $\beta < \dim(E)$ a probability measure $\mu \in \mathcal{P}(E)$, such that

$$\mu(\mathcal{B}(x,r)) \leq Cr^\beta, \qquad \forall x \in \mathbb{R}^d,\ r \in (0,1).$$

But if D denotes the diameter of E,

$$\begin{aligned}
\mathsf{Energy}_\gamma(\mu) &= \sum_{j=0}^\infty \iint\limits_{2^{-j-1}D \leq |x-y| \leq 2^{-j}D} |x-y|^{-\gamma}\, \mu(dx)\, \mu(dy) \\
&\leq \sum_{j=0}^\infty 2^{(j+1)\gamma} D^{-\gamma} \sup_{x \in \mathbb{R}^d} \mu(\mathcal{B}(x, 2^{-j}D)) \\
&\leq C 2^\gamma D^{\beta-\gamma} \sum_{j=0}^\infty 2^{j\gamma} 2^{-j\beta},
\end{aligned}$$

which sums if $\gamma < \beta$. Thus, we have shown that for all $\gamma < \dim(E)$, $\mathsf{Cap}_\gamma(E) > 0$, i.e., $\dim_c(E) \geq \gamma$ for all $\gamma < \dim(E)$, which completes the proof. $\square$

2 The Brownian Curve

Next, we roll up our sleeves and compute the Hausdorff dimension of a few assorted and interesting random fractals that arise from Brownian considerations. Our goal is to illustrate the methods and ideas rather than the final word on this subject.

Throughout, $B = \{B_t;\ t \geq 0\}$ denotes Brownian motion in $\mathbb{R}^d$. That is, a Gaussian process in $\mathbb{R}^d$ such that $B_0 = 0$, and

$$\mathbb{E}\{B^i_t\} = 0 \qquad\qquad \forall t \geq 0,\ i = 1, \dots, d$$

$$\mathbb{E}\{B^i_s B^j_t\} = \begin{cases} s \wedge t & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Recall also that B is a strong Markov process. Then, we have already shown that

$$B \text{ hits points iff } d = 1\text{, i.e., } \exists t > 0 :\ B_t = 0 \iff d = 1.$$

This follows from our proof of Lévy's theorem (Theorem 3.1, Lecture 2.) In particular, note that when $d = 1$, the Brownian curve has full Lebesgue measure, and also full dimension. On the other hand, when $d \geq 2$, the Brownian curve has zero Lebesgue measure, despite the following result.

Theorem 2.1 *If B denotes d-dimensional Brownian motion, where $d \geq 2$, $\dim B(\mathbb{R}_+) = 2$, a.s.*

Proof We do this in two parts. First, we show that $\dim B(\mathbb{R}_+) \leq 2$ (*the upper bound*), and then we show that $\dim B(\mathbb{R}_+) \geq 2$ (*the lower bound*). In any event, recall that $d \geq 2$.

Proof of the upper bound Recall from Theorem 1.2, Lecture 3, that for any interval $I \subset \mathbb{R}^d$,

$$\mathbb{P}\{B[1,2] \cap I \neq \varnothing\} \leq c\kappa(|I|),\ \text{where } \kappa(\varepsilon) = \begin{cases} \varepsilon^{d-2}, & \text{if } d \geq 3 \\ \ln_+\left(\frac{1}{\varepsilon}\right), & \text{if } d = 2 \end{cases}.$$

A careful inspection of the proof shows that the constant c depends only on M, as long as $I \subseteq [-M, M]^d$. Consider $I_1, \dots, I_{n^d}$ cubes of side $\frac{1}{n}$, such that (i) $I_i^\circ \cap I_j^\circ = \varnothing$ if $i \neq j$; and (ii) $\cup_{j=1}^{n^d} I_j = [0,1]^d$. Based on these, define

$$E_j = \begin{cases} I_j, & \text{if } I_j \cap B[1,2] \neq \varnothing \\ \varnothing, & \text{otherwise} \end{cases}.$$

Note that $E_1, \dots, E_{n^d}$ is a $(\frac{1}{n})$-cover of $B[1,2] \cap [0,1]^d$. Thus,

$$\mathcal{H}_\alpha\big(B[1,2] \cap [0,1]^d\big) \leq \liminf_{n\to\infty} \sum_{j=1}^{n^d} n^{-\alpha} \mathbf{1}_{\{I_j \cap B[1,2] \neq \varnothing\}}.$$

Consequently, as long as $\alpha > 2$,

$$\mathbb{E}\big\{\mathcal{H}_\alpha\big(B[1,2] \cap [0,1]^d\big)\big\} \leq c \liminf_{n\to\infty} \sum_{j=1}^{n^d} n^{-\alpha} \kappa\big(\tfrac{1}{n}\big) = c \liminf_{n\to\infty} n^{d-\alpha} \kappa\big(\tfrac{1}{n}\big) = 0.$$

In particular, $\dim(B[1,2]\cap[0,1]^d)\le 2$, a.s. Similarly, $\dim(B[a,b]\cap[-n,n]^d)\le 2$, a.s. for any $0<a<b$ and $n>0$. Let $n\uparrow\infty$, $a\downarrow 0$ and $b\uparrow\infty$, all along rational sequences to deduce that $\dim B(\mathbb{R}_+)\le 2$, a.s. This uses the easily verified fact that whenever $A_1\subseteq A_2\subseteq\cdots$ are compact, and if $\mathcal{H}_\alpha(A_j)=0$, then $\mathcal{H}_\alpha(\cup_j A_j)=0$.

Proof of the lower bound For the converse, we will show that $\dim B[1,2]\ge 2$, and do this by appealing to Frostman's theorem (Theorem 1.1, Lecture 7). To do so, we need to define a probability, or at least a finite, measure on the Brownian curve. The most natural measure that lives on the curve of $\{B_s; 1\le s\le 2\}$ is the occupation measure:

$$\mathbb{O}(E)=\int_1^2 \mathbf{1}_{\{B_s\in E\}}\,ds.$$

With this in mind, note that for any $\alpha>0$,

$$\mathsf{Energy}_\alpha(\mathbb{O})=\iint|x-y|^{-\alpha}\,\mathbb{O}(dx)\,\mathbb{O}(dy)=\int_1^2\int_1^2|B_s-B_t|^{-\alpha}\,ds\,dt.$$

By Frostman's theorem, it suffices to show that $\mathbb{E}\{\mathsf{Energy}_\alpha(\mathbb{O})\}<+\infty$ for all $0<\alpha<2$. But this is easy. Indeed, note that

$$\mathbb{E}\big\{\mathsf{Energy}_\alpha(\mathbb{O})\big\}=2\int_1^2\int_s^2\mathbb{E}\big\{|B_{t-s}|^{-\alpha}\big\}\,ds\,dt=2\int_1^2\int_s^2|t-s|^{-\frac{\alpha}{2}}\,ds\,dt\times\mathbb{E}\{|Z|^{-\alpha}\},$$

where Z is a d-dimensional vector of i.i.d. standard normals. Since $\alpha<2$, the double integral is finite. It suffices to show that $\mathbb{E}\{|Z|^{-\alpha}\rangle<+\infty$. But

$$\begin{aligned}\mathbb{E}\{|Z|^{-\alpha}\}&=\int_0^\infty\mathbb{P}\{|Z|^{-\alpha}>\lambda\}\,d\lambda\\&\le 1+\int_1^\infty\mathbb{P}\{|Z|^{-\alpha}>\lambda\}\,d\lambda\\&=1+\alpha\int_0^1\mathbb{P}\{|Z|<u\}u^{-\alpha-1}\,du && (u=\lambda^{-\frac{1}{\alpha}})\\&=1+\alpha\int_0^1\big[\mathbb{P}\{|Z_1|\le u\}\big]^d u^{-\alpha-1}\,du.\end{aligned}$$

But $\mathbb{P}\{|Z_1|\le u\}=(2\pi)^{-\frac12}\int_{-u}^u e^{-\frac12\lambda^2}\,d\lambda\le u$. Hence, using $d\ge 2$,

$$\mathbb{E}\{|Z|^{-\alpha}\}\le 1+\alpha\int_0^1 u^{d-\alpha-1}\,du\le 1+\alpha\int_0^1 u^{1-\alpha}\,du,$$

which is finite, as promised. $\square$

Lecture 9

Potential Theory of Brownian Motion and Stable Processes

1 Transient Brownian Motion

Classically, probabilistic potential theory has been concerned with connections between Newtonian (and more general potentials) and hitting probabilities for Markov processes. This is a rich theory that we have already been introduced to in earlier lectures in the following form: d-dimensional Brownian motion hits points iff $d = 1$. In fact, we did this by going alot further. Namely, we estimated the hitting probability of a small ball, where by hitting probability we mean something like the quantity $\mathbb{P}\{B[1,2] \cap \mathcal{B}(x, \varepsilon) \neq \varnothing\}$. The classical connections run deep: harmonic and excessive functions, removable singularity for various elliptic PDE's on domains, etc.

To see immediate connections between Brownian motion and Bessel–Riesz potentials (here, Newtonian potentials), consider Brownian motion in $\mathbb{R}^d$ where $d \geq 3$, and define the potential operator $\mathcal{U}$ as

$$\mathcal{U}f(x) = \mathbb{E}_x\Big\{ \int_0^\infty f(B_s)\, ds \Big\} = \mathbb{E}\Big\{ \int_0^\infty f(B_s + x)\, ds \Big\}.$$

We compute this as follows: for all measurable $f : \mathbb{R}^d \to \mathbb{R}_+$,

$$\begin{aligned} \mathcal{U}f(x) &= \int_{\mathbb{R}^d} f(y) \int_0^\infty \frac{e^{-\frac{1}{2s}\|x-y\|^2}}{(2\pi s)^{\frac{d}{2}}}\, ds\, dy \\ &= \frac{\Gamma(\frac{d}{2} - 1)}{2\pi^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{f(y)}{\|x-y\|^{d-2}}\, dy. \end{aligned}$$

Thus, if $\mathfrak{G}f$ denotes the $(d-2)$-dimensional Newtonian (and/or Bessel–Riesz) potential of f defined by $\mathfrak{G}f(x) = \int f(y)|x-y|^{2-d}\, dy$, it follows that $\mathcal{U}f \asymp \mathfrak{G}f$.

Now, let $E \subset \mathbb{R}^d$ denote a compact set whose hitting time is $\tau(E) = \inf\{s \geq 0 : \; B_s \in E\}$, and consider a sub-probability density $f : \mathbb{R}^d \to \mathbb{R}_+$ such that $f = 0$ off of E (OK, provided that E° has positive Lebesgue's measure.) Then,

$$\begin{aligned} \mathcal{U}f(x) &= \mathbb{E}_x\Big\{ \int_{\tau(E)}^\infty f(B_s)\, ds;\; \tau(E) < +\infty \Big\} \\ &= \mathbb{E}_x\Big\{ \int_0^\infty f(B'_s + B_{\tau(E)})\, ds;\; \tau(E) < +\infty \Big\}, \end{aligned}$$

where B' is an independent copy of B. This holds by the strong Markov property. Hence,

$$\begin{aligned} \mathcal{U}f(x) &= \mathbb{E}_x\big\{ \mathcal{U}f(B_{\tau(E)});\; \tau(E) < +\infty \big\} \\ &= \mathbb{E}_x\big\{ \mathcal{U}f(B_{\tau(E)}) \,\big|\, \tau(E) < +\infty \big\} \cdot \mathbb{P}_x\{\tau(E) < +\infty\} \\ &= \int_{\mathbb{R}^d} \mathcal{U}f(x)\mu(dx) \cdot \mathbb{P}_x\{\tau(E) < +\infty\}, \end{aligned}$$

where $\mu(A) = \mathbb{P}_x\{B_{\tau(E)} \in A \,|\, \tau(E) < \infty\}$. Since E° has positive Lebesgue's measure, and since B has continuous samples, $\mu \in \mathcal{P}(E)$, and the above holds for all pdf's f on E. Clearly, $\mu(dx) << dx$. Suppose that E is nice enough that $f = d\mu/dx$ is a probability density function. Note, then, that $\mathcal{U}f(x) = c\int_{\mathbb{R}^d} \|x-y\|^{2-d}\, \mu(dy)$ and $\int \mathcal{U}f\, d\mu = c\iint \|x-y\|^{2-d}\, \mu(dx)\, \mu(dy)$ (for the same c), which yields:

$$\mathbb{P}_x\{\tau(E) < +\infty\} \asymp \frac{\int_{\mathbb{R}^d} \|x-y\|^{2-d}\, \mu(dy)}{\mathsf{Energy}_{d-2}(\mu)}.$$

Next, suppose that the starting point x is strictly outside E, so that for all $y \in E$, $\|x-y\| \asymp 1$. This shows that

$$\mathbb{P}_x\{B(\mathbb{R}_+) \cap E \neq \varnothing\} \asymp \frac{1}{\mathsf{Energy}_{d-2}(\mu)},$$

for some probability measure $\mu \in \mathcal{P}(E)$. In particular,

$$\mathbb{P}_x\{B(\mathbb{R}_+) \cap E \neq \varnothing\} \leq c\mathsf{Cap}_{d-2}(E). \tag{1.1}$$

It is relatively easy to argue that, in fact, the above is essentially optimal. Indeed, note that if $J(h) = \int_0^\infty h(B_s)\, ds$, when h is any pdf on E,

$$\begin{aligned} \mathbb{E}_x\{J(h)\} &= \int_{\mathbb{R}^d} \int_0^\infty h(y) \frac{e^{-\frac{1}{2s}\|x-y\|^2}}{(2\pi s)^{\frac{d}{2}}}\, ds\, dy \\ &\geq c, \end{aligned}$$

since $\sup_{y \in E} \|x-y\| \leq C$ in the above, and since $\int h\, dy = 1$. On the other hand, similar calculations reveal that $\mathbb{E}_x\{|J(h)|^2\} \leq C\mathsf{Energy}_{d-2}(h)$. Thus, by the Paley-Zygmund inequality,

$$\mathbb{P}_x\{\tau(E) < +\infty\} \geq c\Big[\inf_{h=\text{pdf on } E} \mathsf{Energy}_{d-2}(h) \Big]^{-1}.$$

The above describes an absolutely continuous capacity that can be shown to coincide with $\mathsf{Cap}_{d-2}(E)$. Modulo this last step and the part about "E being nice", we have shown the following which is part of a greater theorem of S. Kakutani.

Theorem 1.1 (Kakutani) *When $d \geq 3$, Brownian motion in $\mathbb{R}^d$ can hit a compact set E iff $\mathsf{Cap}_{d-2}(E) > 0$.*

Combining this with Frostman's theorem, we get:

$$\begin{aligned} \dim(E) > d-2 &\implies B \text{ can hit } E \\ \dim(E) < d-2 &\implies B \text{ cannot hit } E. \end{aligned} \tag{1.2}$$

For $d = 1, 2$, the above proof breaks down, since our definition of $\mathcal{U}f$ is typically infinite; this is due to the neighborhood recurrence of Brownian motion in $\mathbb{R}^d$, $d \leq 2$. However, the basic principle is still correct, as long as we do something about times near $+\infty$. The classical way to do this is to introduce an independent mean 1 exponential random variable $\mathfrak{e}$ and define the 1-potential operator

$$\mathcal{U}_1 f(x) = \mathbb{E}_x \Big\{ \int_0^{\mathfrak{e}} f(B_s)\, ds \Big\}.$$

The reason for the choice of the exponential law is that B stopped at $\mathfrak{e}$ is still a strong Markov process. Also, check that

$$\mathcal{U}_1 f(x) = \mathbb{E}_x \Big\{ \int_0^{\infty} e^{-s} f(B_s)\, ds \Big\}.$$

Now, proceed as in the proof of Kakutani's theorem above, but pay attention to the cases $d = 1$ and $d = 2$ separately; we also need to replace $\mathcal{U}f$ by $\mathcal{U}_1 f$ everywhere. This will lead to the complete

Theorem 1.2 (Kakutani's Theorem) *Brownian motion in any dimension d hits a compact set E iff $\mathsf{Cap}_{d-2}(E) > 0$.*

The result is trivial when $d = 1$ and $\mathsf{Cap}_{-1}(E) = 1$ for all E. So, the content is in dimension 2, where Brownian motion hits only (and all) compact sets of positive logarihtmic capacity. In particular, Eq. (1.2) holds in all dimensions $d \geq 2$.

2 Additive Brownian Motion

Suppose X and Y are two independent d-dimensional Brownian motions. We have already encountered the problem of deciding when $X(\mathbb{R}_+) \cap Y(\mathbb{R}_+) \neq \varnothing$. One way to interpret this is by the following identification:

$X(\mathbb{R}_+) \cap Y(\mathbb{R}_+) \neq \varnothing$ the origin in $\mathbb{R}^d$. One can ask, more generally, what types of sets can Z hit? By symmetry, we can rewrite Z (in law) as

$$Z_{s,t} = X_s + Y_t, \qquad \forall s, t \geq 0,$$

and call it *additive Brownian motion* (with 2 parameters). More generally, we can define N-parameter additive Brownian motion Z as

$$Z_t = \sum_{j=1}^{N} B^j_{t_j}, \qquad \forall t \in \mathbb{R}^N_+,$$

where $B^1, \ldots, B^N$ are iid d-dimensional Brownian motions. We can refer to Z as (N, d)-additive Brownian motion to keep the dimensions straight.

Theorem 2.1 (Hirsch and Song; Kh and Shi) *(N, d)-additive Brownian motion can hit a compact set E iff $\mathsf{Cap}_{d-2N}(E) > 0$.*

Sketch of Proof We will sketch an argument that shows that if Z is (N, d)-additive Brownian motion, $\mathbb{P}\{Z[1, 2]^N \cap E \neq \varnothing\} > 0$ iff $\mathsf{Cap}_{d-2N}(E) > 0$. Going from $[1, 2]^N$ to $\mathbb{R}^N_+$ is standard, since the same argument shows that for any cube $[a, b] = \prod_{j=1}^N [a_j, b_j]$ with $a_j > 0$, $\mathbb{P}\{Z[a, b] \cap E \neq \varnothing\} > 0$ iff $\mathsf{Cap}_{d-2N}(E) > 0$. Thus, we can let $a_j \downarrow 0$ and $b_j \uparrow \infty$ to finish. With this in mind, consider

$$J(f) = \int_{[1,2]^N} f(Z_s)\, ds.$$

Then, one shows that if f_ε is a pdf on the ε-enlargement E^ε of E, $\inf_{\varepsilon>0} \mathbb{E}\{J(f_\varepsilon)\} \geq c$ and $\mathbb{E}\{|J(f_\varepsilon)|^2\} \leq c\mathsf{Energy}_{d-4}(f_\varepsilon)$. In both of these estimates, c depends on the outer radius of E only. But $\{J(f_\varepsilon) > 0\}$ implies $\{Z[1, 2]^N \cap E^\varepsilon \neq \varnothing\}$. Hence, $\mathbb{P}\{Z[1, 2]^N \cap E^\varepsilon \neq \varnothing\} \geq c\mathsf{Cap}_{d-2N}(E^\varepsilon) \geq c\mathsf{Cap}_{d-2N}(E)$. Let $\varepsilon \to 0$ and use the compactness of E, together with the continuity of Z.

For the harder converse, we will prove that $\mathbb{P}\{Z[1, \frac{3}{2}]^N \cap E \neq \varnothing\} > 0$ implies $\mathsf{Cap}_{d-2N}(E) > 0$. Henceforth, we assume

$$\mathbb{P}\{Z[1, \tfrac{3}{2}]^N \cap E \neq \varnothing\} > 0, \tag{2.1}$$

and let $\mathcal{F}_t$ denote the σ-field generated by $\{Z_r;\ r \preccurlyeq t\}$. Note that $\mathcal{F}$ is commuting, and consider the N-parameter martingale $M(f)$ given by

$$M_t(f) = \mathbb{E}\{J(f) \,|\, \mathcal{F}_t\}, \qquad \forall t \in \mathbb{R}^N_+.$$

Clearly, for all $t \in [1, \frac{3}{2}]^N$,

$$
\begin{aligned}
M_t(f) &\geq \int_{t \preccurlyeq s \preccurlyeq (2,\ldots,2)} \mathbb{E}\{f(Z_s) \,|\, \mathcal{F}_t\} \, ds \\
&= \int_{t \preccurlyeq s \preccurlyeq (2,\ldots,2)} \mathbb{E}\{f(Z'_{s-t} + Z_t \,|\, \mathcal{F}_t\} \, ds,
\end{aligned}
$$

where Z' is an independent copy of Z. Therefore,

$$
M_t(f) \geq \int_{[0,\frac{1}{2}]^N} \mathbb{E}\{f(Z'_r + Z_t) \,|\, \mathcal{F}_t\} \, dr, \qquad \forall t \in [1, \tfrac{3}{2}] \cap \mathbb{Q}_+^N.
$$

Now, suppose $T^\varepsilon \in \mathbb{Q}_+^N \cup \{\infty\}$ is any random variable such that

(i) $T^\varepsilon \neq \infty$ iff $Z[1, \frac{3}{2}]^N \cap E^\varepsilon \neq \varnothing$ where E^ε is the closed ε-enlargement of E;

(ii) on $\{T^\varepsilon \neq \infty\}$, $Z_{T^\varepsilon} \in E^\varepsilon$.

The previous inequality for $M_t(f)$ implies,

$$
M_{T^\varepsilon} \mathbf{1}_{\{T^\varepsilon \neq \infty\}} \geq \int_{[0,\frac{1}{2}]^N} \mathbb{E}\{f(Z'_r + Z_{T^\varepsilon}) \,|\, Z_{T^\varepsilon}, T^\varepsilon \neq \infty\} \, dr \cdot \mathbf{1}_{\{T^\varepsilon \neq \infty\}}.
$$

Define, for simplicity, the operator $\mathcal{V}f(x) = \int_{[0,\frac{1}{2}]^N} \mathbb{E}\{f(Z_r + x)\} \, dr$, to see that

$$
M_{T^\varepsilon} \mathbf{1}_{\{T^\varepsilon \neq \infty\}} \geq \mathcal{V}f(Z_{T^\varepsilon}) \cdot \mathbf{1}_{\{T^\varepsilon \neq \infty\}}.
$$

Thanks to Eq. (2.1), $\mu_\varepsilon \in \mathcal{P}(E^\varepsilon)$, where

$$
\mu_\varepsilon(\bullet) = \mathbb{P}\{Z_{T^\varepsilon} \in \bullet \,|\, T^\varepsilon \neq \infty\}.
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\{\sup_{t \in \mathbb{Q}_+^N} |M_t(f)|^2\} &\geq \mathbb{E}\big\{ \big[\mathcal{V}f(Z_{T^\varepsilon})\big]^2 \cdot \mathbf{1}_{\{T^\varepsilon \neq \infty\}} \big\} \\
&= \mathbb{E}\big\{ \big[\mathcal{V}f(Z_{T^\varepsilon})\big]^2 \,\big|\, T^\varepsilon \neq \infty \big\} \cdot \mathbb{P}\{T^\varepsilon \neq \infty\} \\
&\geq \big[\mathbb{E}\{\mathcal{V}f(Z_{T^\varepsilon}) \,|\, T^\varepsilon \neq \infty\}\big]^2 \cdot \mathbb{P}\{T^\varepsilon \neq \infty\} \\
&= \Big[\int \mathcal{V}f(x) \, \mu_\varepsilon(dx) \Big]^2 \cdot \mathbb{P}\{T^\varepsilon \neq \infty\}.
\end{aligned}
$$

By Cairoli's inequality, and by the mentioned second moment estimate for $J(f)$, $\mathbb{E}\{\sup_{t \in \mathbb{Q}_+^N} |M_t(f)|^2\} \leq 4^N \sup_t \mathbb{E}\{|M_t(f)|^2\} \leq c\mathsf{Energy}_{d-2N}(f)$. This leads us to

$$
c\mathsf{Energy}_{d-2N}(f) \geq \Big[\int \mathcal{V}f(x) \, \mu_\varepsilon(dx) \Big]^2 \cdot \mathbb{P}\{T^\varepsilon \neq \infty\}.
$$

Now, let $\varphi_\eta = (2\eta)^{-d}\mathbf{1}_{\mathcal{B}(0,\eta)}$ be an approximation to the identity, and let $f = \mu_\varepsilon \star \varphi_\eta$. Then, f is a pdf on $E^{\eta+\varepsilon}$ and converges weakly to μ_ε as $\eta \downarrow 0$. One can show that $\mathsf{Energy}_{d-2N}(f) \to \mathsf{Energy}_{d-2N}(\mu_\varepsilon)$, as $\eta \to 0$, as well. We now wish to at least argue why $\int \mathcal{V}f\, d\mu_\varepsilon \to \mathsf{Energy}_{d-2N}(\mu_\varepsilon)$ as $\eta \to 0$, as well. If so, we can deduce that

$$\mathbb{P}\{T^\varepsilon \neq \infty\} \leq c \div \mathsf{Energy}_{d-2N}(\mu_\varepsilon) \leq c\mathsf{Cap}_{d-2N}(E^\varepsilon),$$

which is our result but with E replaced by E^ε. Letting $\varepsilon \to 0$ will lead everything to converge to their proper limit. That is, $\mathbb{P}\{T^\varepsilon \neq \infty\} \to \mathbb{P}\{Z[1, \frac{3}{2}]^N \cap E \neq \varnothing\}$ (easy to see this), and $\mathsf{Cap}_{d-2N}(E^\varepsilon) \to \mathsf{Cap}_{d-2N}(E)$ (harder to show.) It remains to identify the limit as $\eta \to 0$ of $\int \mathcal{V}f\, d\mu_\varepsilon$. But

$$\begin{aligned}\int \mathcal{V}f\, d\mu_\varepsilon &= \int_{\mathbb{R}^d} \int_{[0,\frac{1}{2}]^N} \mathbb{E}\{f(Z_t + x)\}\, dt\, \mu_\varepsilon(dx) \\ &= \iint \int_{[0,\frac{1}{2}]^N} p_t(x-y)\, dt\, f(y)dy\, \mu_\varepsilon(dx),\end{aligned}$$

where $p_t(x) = (2\pi \sum_{i=1}^N t_i)^{-\frac{d}{2}} \exp(-\frac{\|x\|^2}{2\sum_{i=1}^N t_i})$. Now, let $\eta \to 0$ to see that $f(y)dy \simeq \mu_\varepsilon(dy)$, which does the job. $\square$

3 Additive Stable Processes

Recall that an $\mathbb{R}^d$-valued process $X = \{X_t;\ t \geq 0\}$ is an *isotropic stable process* of index $\alpha \in (0, 2]$ if

St-1. For each t, $\mathbb{E}\{e^{i\xi\cdot X_t}\} = \exp(-\frac{1}{2}\|\xi\|^\alpha)$;

St-2. $X_0 = 0$, a.s.; and

St-3. for each $t \geq 0$, $s \mapsto X_{t+s} - X_t$ is a copy of X that is independent of $\{X_u;\ u \leq t\}$.

The condition that $\alpha \in (0, 2]$ is forced on us by the above conditions, and when $\alpha = 2$, X is Brownian motion.

To perform potential-theoretic calculations, we only need estimates for the pdf of X_t; all else is done as in Brownian motion. Here are the requisite facts. All can be found in MPP; Chapters 8, and 10, together with proofs. Throughout, $p_t(x) = \mathbb{P}\{X_t \in dx\}/dx$.

By the Fourier inversion formula,

$$p_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} e^{-\frac{1}{2}\|\xi\|^\alpha}\, d\xi.$$

This can be used to show to that

- $(t, x) \mapsto p_t(x)$ is continuous and strictly positive on compact subsets of $(0, \infty) \times \mathbb{R}^d$;

- **(scaling)** $p_t(x) = t^{-\frac{d}{\alpha}} p_1(x/t^{\frac{1}{\alpha}})$;
- **(unimodality)** $p_t(x) \le p_t(0) = Ct^{-\frac{d}{\alpha}}$, for all $x \in \mathbb{R}^d$;
- **(isotropy)** $p_t(x) = p_t(y)$ if $\|x\| = \|y\|$;
- **(Blumenthal-Getoor's asymptotics)** as $\|z\| \to \infty$, $p_1(z) \sim C\|z\|^{-(d+\alpha)}$.

An N-parameter *additive stable process* Z is defined by

$$Z_t = \sum_{j=1}^{N} X^j_{t_j}, \qquad \forall t \in \mathbb{R}^N_+,$$

where $X^1, \ldots, X^N$ are i.i.d. isotropic stable processes of index $\alpha \in (0, 2]$. Then, the above together with the Brownian methods outlined earlier, can be used in conjunction to prove the following:

Theorem 3.1 (Hirsch and Song; MPP Ch. 11) *If Z denotes an (N, d)-additive stable process of index $\alpha \in (0, 2]$, then Z hits a compact set $E \subset \mathbb{R}^d$ iff $\mathsf{Cap}_{d-\alpha N}(E) > 0$.*

This is attractive, since it relates Cap_β to a stochastic process for every $\beta \in \mathbb{R}_+$; when $N = 1$, this is classical, but only connects probability to Cap_β where $\beta \in [d-2, d]$. One can use this to show also the following.

Theorem 3.2 (MPP Ch. 11) *Suppose Z is as above. Then, with probability one,*

$$\dim Z(\mathbb{R}^N_+) = d \wedge \alpha N.$$

To prove this, we only need the following, which is, in fact, a consequence of Theorem 3.1:

$$\mathbb{P}\{Z[1,2]^N \cap \mathcal{B}(x, \varepsilon) \neq \varnothing\} \asymp C \times \begin{cases} \varepsilon^{d-\alpha N}, & \text{if } d > \alpha N \\ \{\log(1/\varepsilon)\}^{-1}, & \text{if } d = \alpha N \\ 1, & \text{if } d < \alpha N \end{cases}.$$

In fact, the constants depend only on M, and the above holds uniformly for all $x \in [-M, M]^d$.

4 Application to Stochastic Codimension

The preceeding has a remarkable consequence about a large class of random sets. We say that a random set $X \subset \mathbb{R}^d$ has *codimension* β, if β is the critical number such that for all compact sets $E \subset \mathbb{R}^d$ with $\dim(E) > \beta$, $\mathbb{P}\{X \cap E \neq \varnothing\} > 0$, while for all compact sets $F \subset \mathbb{R}^d$ with $\dim(F) < \beta$, $\mathbb{P}\{X \cap F \neq \varnothing\} = 0$. The notion of codimension was coined in this way in Kh-Shi '99, but the essential idea has been around in the works of Taylor '65, Lyons '99, Peres '95, $\cdots$

When it does exist, the codimension of a random set is a nonrandom number.

Warning: Not all random sets have a codimension.

The following is a fancy example for the above.

Theorem 4.1 (Kh-Peres-Xiao) *Let B denote Brownian motion, and consider*

$$F(\lambda) = \left\{ t \geq 0 : \limsup_{\varepsilon \to 0} \frac{|B_{t+\varepsilon} - B_t|}{\sqrt{2\varepsilon \log(1/\varepsilon)}} = \lambda \right\}.$$

Then, for any compact set $E \subset \mathbb{R}_+$, $\mathbb{P}\{F(\lambda) \cap E \neq \varnothing\} = 0$ if $\dim_p(E) < \lambda^2$, while it is 1 if $\dim_p(E) > \lambda^2$, where $\dim_p$ denotes "packing dimension."

All that you need to know of packing dimension, here, is that it is not Hausdorff dimension although $\dim_p \geq \dim$. In fact, there are compact sets $E \subset \mathbb{R}_+$ such that $\dim_p(E) = 1$ while $\dim(E) = 0$. Multi-dimensional examples are also possible.

As examples of random sets that *do* have codimension, we mention the following consequence of Theorem 3.1:

Corollary 4.2 *If Z denotes an (N, d)-additive stable process of index $\alpha \in (0, 2]$,* $\text{codim}(Z[1,2]^N) = d - \alpha N$.

We now wish to use Theorem 3.1 to prove the following result. In the present form, it is from MPP Ch. 11, but for $d = 1$, it is from Kh-Shi '99.

Theorem 4.3 (MPP Ch. 11) *If X is a random set in $\mathbb{R}^d$ that has codimension $\beta \in (0, d)$,*

$$\dim(X) = d - \text{codim}(X), \qquad \textit{a.s.}$$

That is, in the best of circumstances,

$$\boxed{\dim(X) + \text{codim}(X) = \text{topological dimension.}}$$

A note of warning: if X is not compact, $\dim(X)$ can be defined by $\sup_{n\geq 1} \dim(\overline{X \cap [-n, n]^d})$.

The proof depends on the following result that can be found in the works of Yuval Peres '95, but with percolation proofs.

Lemma 4.4 (Peres' lemma) *For each $\beta \in (0, d)$, there exists a random set Λ_β, whose codimension is β. Moreover, $\dim(\Lambda_\beta) = d - \beta$, almost surely.*

Proof Let $\Lambda_\beta = Z(\mathbb{R}_+^N)$, where Z is an (N, d)-addtive stable process. The result follows from Corollary 4.2 and Theorem 3.2. $\square$

Proof of Theorem 4.3 By localization, we may assume that X is a.s. compact. Let $\Lambda_\beta = \cup_{i=1}^\infty \Lambda_\beta^i$, where $\Lambda_\beta^1, \Lambda_\beta^2, \ldots$ are iid copies of the sets in Peres' lemma, and are all totally independent of our random set X. Then, by Peres' lemma and by the lemma of Borel–Cantelli,

$$\mathbb{P}\{\Lambda_\beta \cap X \neq \varnothing \,|\, X\} = \begin{cases} 0, & \text{on } \{\dim(X) < \beta\} \\ 1, & \text{on } \{\dim(X) > \beta\} \end{cases}.$$

On the other hand, by the very definition of codimension,

$$\mathbb{P}\{\Lambda_\beta \cap X \neq \varnothing \,|\, \Lambda_\beta\} = \begin{cases} 0, & \text{if codim}(X) > d - \beta = \dim(\Lambda_\beta) \\ > 0, & \text{if codim}(X) < d - \beta \end{cases}.$$

Take expectations of the last two displays to see that for any $\beta \in (0, d)$,

$$\begin{aligned} \text{codim}(X) < d - \beta &\implies \dim(X) \geq \beta, \text{ a.s.} \\ \text{codim}(X) > d - \beta &\implies \dim(X) \leq \beta, \text{ a.s.} \end{aligned}$$

This easily proves our theorem. □

Lecture 10

Brownian Sheet and Kahane's Problem

The (N, d)-Brownian sheet $B = \{B_t;\, t \in \mathbb{R}_+^N\}$ is an $\mathbb{R}^d$-valued, N-parameter Gaussian process with i.i.d. coordinate processes, $B^1, \ldots, B^d$, each of which has the covariance:

$$\mathbb{E}\{B_s^1 B_t^1\} = \prod_{\ell=1}^N (s_\ell \wedge t_\ell), \qquad \forall s, t \in \mathbb{R}_+^N.$$

For instance, consider $N = 2$ and write $B_{u,v}$ for the sheet. Then,

- for each fixed u, $v \mapsto u^{-\frac{1}{2}} B_{u,v}$ is a Brownian motion;
- for each fixed v, $u \mapsto v^{-\frac{1}{2}} B_{u,v}$ is a Brownian motion;
- for each $c > 0$ fixed, the process B_{ce^{-v}, e^v} has d i.i.d. coordinates each of which has covariance,

  $$\mathbb{E}\{B^1_{ce^{-v}, e^v} B^1_{ce^{-u}, e^u}\} = c^2 \exp(-|u - v|).$$

  That is, $v \mapsto B_{ce^{-v}, e^v}$ is an Ornstein–Uhlenbeck process.

You can also find all manners of time-changes of Brownian motion within Brownian sheet.

1 Local Structure and Potential Theory

1.1 Independent Increments

If $t \succcurlyeq s$, $B_t - B_s$ is independent of $\{B_u;\, u \preccurlyeq s\}$. Since all is Gaussian, we check this by computing covariances, all the time assuming that $d = 1$, viz., for all $u \preccurlyeq s$,

$$\mathbb{E}\{(B_t - B_s) B_u\} = \prod_{\ell=1}^N (t_\ell \wedge u_\ell) - \prod_{i=1}^N (s_\ell \wedge u_\ell) = 0.$$

1.2 Incremental Laws

Whenever $t \succcurlyeq s$, $B_t - B_s$ is a vector of d i.i.d. centered Gaussians with variance

$$\mathbb{E}\{(B_t^1 - B_s^1)^2\} = \prod_{j=1}^{N} t_j + \prod_{k=1}^{N} s_k - 2\prod_{i=1}^{N}(t_i \wedge s_i) = \prod_{j=1}^{N} t_j - \prod_{k=1}^{N} s_k.$$

In particular, if s is fixed, the above, for all $t \succcurlyeq s$ that are close to s, is $\simeq \sum_{j=1}^{k} |t_j - s_j|$. This is done by Taylor expansions. In particular, locally, $B_t - B_s$ has the same law as (N, d) additive Brownian motion.

Armed with the above estimate, one can then prove

Theorem 1.1 (Kh and Shi) *If E is a compact set in $\mathbb{R}^d$, $\mathbb{P}\{B(\mathbb{R}_+^N) \cap E \neq \varnothing\} > 0$ iff $\mathsf{Cap}_{d-2N}(E) > 0$. In fact, for any $M > 0$, there exists c_1 and c_2, such that for all compact $E \subset [-M, M]^d$,*

$$c_1 \mathsf{Cap}_{d-2N}(E) \leq \mathbb{P}\{B[1,2]^N \cap E \neq \varnothing\} \leq c_2 \mathsf{Cap}_{d-2N}(E).$$

An immediate consequence of this is that $\mathrm{codim} B(\mathbb{R}_+^N) = d - 2N$. Thus, essentially by Theorem 4.3 of Lecture 9, we have

Corollary 1.2 *With probability one,*

$$\dim B(\mathbb{R}_+^N) = 2N \wedge d.$$

(Essentially refers to the fact that the $2N \geq d$ case needs to be handled separately, but in the latter cases, it is not hard to show directly that the dimension is d.)

2 Kahane's Problem

We come to the last portion of these lectures, which is on a class of problems that I call Kahane's problem, due to the work of J.-P. Kahane in this area.

Kahane's problem for a random field X is: "*when does $X(E)$ have positive Lebesgue's measure?*" I will work the details out for Brownian motion, where things are easier. The problem for the Brownian sheet was partly solved by Kahane (cf. his '86 book) and completely solved by Kh. '99 in case $N = 2$. Recent work of Kh. and Xiao '01 has completed the solution to Kahane's problem and a class of related problems, and we hope to write this up at some point. Here is the story for Brownian motion, where we work things out more or less completely. The story for Brownian sheet is more difficult, and I will say some words about the details later.

Theorem 2.1 (Kahane; Hawkes) *If B denotes Brownian motion in $\mathbb{R}$, and if $E \subset \mathbb{R}_+$ is compact, then*

$$\mathbb{E}\{|B(E)|\} > 0 \iff \mathsf{Cap}_{\frac{d}{2}}(E) > 0.$$

You can interpret this as a statement about hitting proabilities for the level sets of Brownian motion, viz.,

$$\int_{\mathbb{R}^d} \mathbb{P}\{B^{-1}\{a\} \cap E \neq \varnothing\}\, da > 0 \iff \mathsf{Cap}_{\frac{d}{2}}(E) > 0.$$

I will prove the following for Brownian motion. It clearly implies the above theorem upon integration.

Theorem 2.2 *Suppose $E \subset [1, 2]$ is compact, and fix $M > 0$. Then, there exists c_1 and c_2 such that for all $|a| \leq M$,*

$$c_1 \mathsf{Cap}_{\frac{1}{2}}(E) \leq \mathbb{P}\{a \in B(E)\} \leq c_2 \mathsf{Cap}_{\frac{1}{2}}(E).$$

Proof Without loss of any generality, we may and will assume that $E \subseteq [0, 1]$.

For any $\mu \in \mathcal{P}(E)$ and for all $a \in \mathbb{R}$, define

$$J^a_\varepsilon(\mu) = (2\varepsilon)^{-1} \int_0^\infty \mathbf{1}_{\{|B_s - a| \leq \varepsilon\}}\, \mu(ds).$$

Then, for every $M > 0$, there exists c such that

$$\begin{aligned} \inf_{\varepsilon \in (0,1)} \inf_{a \in [-M,M]} \mathbb{E}\{J^a_\varepsilon(\mu)\} &\geq c, \text{ and} \\ \sup_{a \in \mathbb{R}} \sup_{\varepsilon \in (0,1)} \mathbb{E}\{|J^a_\varepsilon(\mu)|^2\} &\leq \mathsf{Energy}_{\frac{d}{2}}(\mu). \end{aligned} \tag{2.1}$$

Now, we apply Paley–Zygmund inequality:

$$\begin{aligned} \mathbb{P}\{a \in B(E)\} &\geq \mathbb{P}\{J^a_\varepsilon(\mu) > 0\} \\ &\geq \frac{|\mathbb{E}\{J^a_\varepsilon(\mu)\}|^2}{\mathbb{E}\{|J^a_\varepsilon(\mu)|^2\}} \\ &\geq \frac{c}{\mathsf{Energy}_{\frac{d}{2}}(\mu)}. \end{aligned}$$

Since this holds for all $\mu \in \mathcal{P}(E)$, we obtain the desired lower bound.

Let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the filtration of B and consider the martingale

$$M_t^{a,\varepsilon}(\mu) = \mathbb{E}\{J_\varepsilon^a(\mu) \,|\, \mathcal{F}_t\}, \qquad \forall t \geq 0.$$

Clearly,

$$\begin{aligned}
M_t^{a,\varepsilon}(\mu) &\geq (2\varepsilon)^{-1} \int_{s\geq t} \mathbb{P}\{|B_s - a| \leq \varepsilon \,|\, \mathcal{F}_t\}\, \mu(ds) \cdot \mathbf{1}_{\{|B_t - a| \leq \frac{\varepsilon}{2}\}} \\
&\geq (2\varepsilon)^{-1} \int_{s\geq t} \mathbb{P}\{|B_{s-t}| \leq \tfrac{1}{2}\varepsilon\}\, \mu(ds) \cdot \mathbf{1}_{\{|B_t - a| \leq \frac{\varepsilon}{2}\}} \\
&\geq c\varepsilon^{-1} \int_{s\geq t} \Big(\frac{\varepsilon}{\sqrt{s-t}} \wedge 1\Big)\, \mu(ds) \cdot \mathbf{1}_{\{|B_t - a| \leq \frac{\varepsilon}{2}\}}
\end{aligned}$$

Let $\sigma_\varepsilon = \inf\{s \in E : \, |B_s - a| \leq \frac{1}{2}\varepsilon\}$. This is a stopping time and on $\{\sigma_\varepsilon < \infty\}$,

$$M_{\sigma_\varepsilon}^{a,\varepsilon}(\mu) \geq c\varepsilon^{-1} \int_{s\geq \sigma_\varepsilon} \Big[\frac{\varepsilon}{\sqrt{s-\sigma_\varepsilon}} \wedge 1\Big]\, \mu(ds),$$

since all bounded Brownian martingales are continuous. Now, we choose μ carefully: WLOG $\mathbb{P}\{\sigma_\varepsilon < \infty\} > 0$ which implies that $\mu_\varepsilon \in \mathcal{P}(E)$, where

$$\mu_\varepsilon(\bullet) = \mathbb{P}\{\sigma_\varepsilon \in \bullet \,|\, \sigma_\varepsilon < \infty\}.$$

Thus, by the optional stopping theorem,

$$\begin{aligned}
1 &\geq \mathbb{E}\{M_{\sigma_\varepsilon}^{a,\varepsilon}(\mu_\varepsilon); \sigma_\varepsilon < \infty\} \\
&\geq c\varepsilon^{-1} \iint\limits_{s\geq t} \Big(\frac{\varepsilon}{\sqrt{s-t}} \wedge 1\Big)\, \mu_\varepsilon(ds)\, \mu_\varepsilon(dt) \cdot \mathbb{P}\{\sigma_\varepsilon < \infty\} \\
&\geq \frac{c}{2}\varepsilon^{-1} \iint \Big(\frac{\varepsilon}{\sqrt{s-t}} \wedge 1\Big)\, \mu_\varepsilon(ds)\, \mu_\varepsilon(dt) \cdot \mathbb{P}\{\sigma_\varepsilon < \infty\} \\
&= \frac{c}{2} \iint \Big(\frac{1}{\sqrt{s-t}} \wedge \frac{1}{\varepsilon}\Big)\, \mu_\varepsilon(ds)\, \mu_\varepsilon(dt) \cdot \mathbb{P}\{\sigma_\varepsilon < \infty\}.
\end{aligned}$$

Fix $\delta_0 > 0$ and from the above deduce that for all ε small,

$$1 \geq \frac{c}{2} \iint\limits_{|s-t|\geq\delta_0} |s-t|^{-\frac{1}{2}}\, \mu_\varepsilon(ds)\, \mu_\varepsilon(dt) \cdot \mathbb{P}\{\inf_{t\in E} |B_t - s| \leq \tfrac{1}{2}\varepsilon\}.$$

Let $\varepsilon \to 0$, and envoke Prohorov's theorem to get $\mu \in \mathcal{P}(E)$ such that

$$\mathbb{P}\{a \in B(E)\} \leq \frac{2}{c}\Big[\iint_{|s-t|\geq\delta_0} |s-t|^{-\frac{1}{2}}\,\mu(ds)\,\mu(dt)\Big]^{-1}.$$

Let $\delta_0 \downarrow 0$ to finish. $\square$

To prove the general result, one needs the properties of the process B around a time point t. There are 2^N different notions of 'around', one for each quadrant centered at t, and this leads to 2^N different N-parameter martingales, each of which is a martingale with respect to a commuting filtration, but each filtration is indeed a filtration with respect to a different partial order. The details are complicated enough for $N = 2$ and can be found in my paper in the *Transactions of the AMS* (1999). When $N > 2$, the details are more complicated still and will be written up in the future. I will end with a related

CONJECTURE: Suppose X is an (N, d) symmetric stable sheet with index $\alpha \in (0, 2]$ (see below.) Then, for any compact $E \subset \mathbb{R}^N_+$, $\mathbb{E}\{|X(E)|\} > 0$ iff $\mathsf{Cap}_{\frac{d}{\alpha}}(E) > 0$.

At the moment, this seems entirely out of the reach of the existing theory, but the analogous result for additive stable processes, and much more, holds (joint work with Xiao–will write up later.)

To finish: $\{X_t;\, t \in \mathbb{R}^N_+\}$ is an (N, d) symmetric stable sheet if it has i.i.d. cooridinates and the first coordinate has the representation $X^1_t = \int \mathbf{1}_{\{0\preccurlyeq s\preccurlyeq t\}}\mathbb{X}(ds)$, where $\mathbb{X}$ is a totally scattered random measure such that for every nonrandom measurable $A \subset \mathbb{R}^N_+$, $\mathbb{E}\{\exp[i\xi\mathbb{X}(A)]\} = \exp(-\frac{1}{2}|A|\ \|\xi\|^\alpha)$. (Scattered means that for nonrandom measurable A and A' in $\mathbb{R}^N_+$, if $A \cap A' = \varnothing$, $\mathbb{X}(A)$ and $\mathbb{X}(A')$ are independent.)

Bibliography

Adler, R. J. (1977). Hausdorff dimension and Gaussian fields. *Ann. Probability 5*(1), 145–151.

Adler, R. J. (1980). A Hölder condition for the local time of the Brownian sheet. *Indiana Univ. Math. J. 29*(5), 793–798.

Adler, R. J. (1981). *The Geometry of Random Fields*. Chichester: John Wiley & Sons Ltd. Wiley Series in Probability and Mathematical Statistics.

Adler, R. J. (1990). *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*. Hayward, CA: Institute of Mathematical Statistics.

Adler, R. J. and R. Pyke (1997). Scanning Brownian processes. *Adv. in Appl. Probab. 29*(2), 295–326.

Aizenman, M. and B. Simon (1982). Brownian motion and Harnack inequality for Schrödinger operators. *Comm. Pure Appl. Math. 35*(2), 209–273.

Bakry, D. (1979). Sur la régularité des trajectoires des martingales à deux indices. *Z. Wahrsch. Verw. Gebiete 50*(2), 149–157.

Bakry, D. (1981a). Limites "quadrantales" des martingales. In *Two-index random processes (Paris, 1980)*, pp. 40–49. Berlin: Springer.

Bakry, D. (1981b). Théorèmes de section et de projection pour les processus à deux indices. *Z. Wahrsch. Verw. Gebiete 55*(1), 55–71.

Bakry, D. (1982). Semimartingales à deux indices. *Ann. Sci. Univ. Clermont-Ferrand II Math.* (20), 53–54.

Barlow, M. T. (1988). Necessary and sufficient conditions for the continuity of local time of Lévy processes. *Ann. Probab. 16*(4), 1389–1427.

Barlow, M. T. and J. Hawkes (1985). Application de l'entropie métrique à la continuité des temps locaux des processus de Lévy. *C. R. Acad. Sci. Paris Sér. I Math. 301*(5), 237–239.

Barlow, M. T. and E. Perkins (1984). Levels at which every Brownian excursion is exceptional. In *Seminar on probability, XVIII*, pp. 1–28. Berlin: Springer.

Bass, R. (1987). L_p inequalities for functionals of Brownian motion. In *Séminaire de Probabilités, XXI*, pp. 206–217. Berlin: Springer.

Bass, R. F. (1985). Law of the iterated logarithm for set-indexed partial sum processes with finite variance. *Z. Wahrsch. Verw. Gebiete 70*(4), 591–608.

Bass, R. F. (1995). *Probabilistic Techniques in Analysis*. New York: Springer-Verlag.

Bass, R. F. (1998). *Diffusions and Elliptic Operators*. New York: Springer-Verlag.

Bass, R. F., K. Burdzy, and D. Khoshnevisan (1994). Intersection local time for points of infinite multiplicity. *Ann. Probab. 22*(2), 566–625.

Bass, R. F. and D. Khoshnevisan (1992a). Local times on curves and uniform invariance principles. *Probab. Theory Related Fields 92*(4), 465–492.

Bass, R. F. and D. Khoshnevisan (1992b). Stochastic calculus and the continuity of local times of Lévy processes. In *Séminaire de Probabilités, XXVI*, pp. 1–10. Berlin: Springer.

Bass, R. F. and D. Khoshnevisan (1993a). Intersection local times and Tanaka formulas. *Ann. Inst. H. Poincaré Probab. Statist. 29*(3), 419–451.

Bass, R. F. and D. Khoshnevisan (1993b). Rates of convergence to Brownian local time. *Stochastic Process. Appl. 47*(2), 197–213.

Bass, R. F. and D. Khoshnevisan (1993c). Strong approximations to Brownian local time. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, pp. 43–65. Boston, MA: Birkhäuser Boston.

Bass, R. F. and D. Khoshnevisan (1995). Laws of the iterated logarithm for local times of the empirical process. *Ann. Probab. 23*(1), 388–399.

Bass, R. F. and R. Pyke (1984a). The existence of set-indexed Lévy processes. *Z. Wahrsch. Verw. Gebiete 66*(2), 157–172.

Bass, R. F. and R. Pyke (1984b). Functional law of the iterated logarithm and uniform central limit theorem for partial-sum processes indexed by sets. *Ann. Probab. 12*(1), 13–34.

Bass, R. F. and R. Pyke (1984c). A strong law of large numbers for partial-sum processes indexed by sets. *Ann. Probab. 12*(1), 268–271.

Bass, R. F. and R. Pyke (1985). The space $\mathcal{D}(A)$ and weak convergence for set-indexed processes. *Ann. Probab. 13*(3), 860–884.

Bauer, J. (1994). Multiparameter processes associated with Ornstein-Uhlenbeck semigroups. In *Classical and Modern Potential Theory and Applications (Chateau de Bonas, 1993)*, pp. 41–55. Dordrecht: Kluwer Acad. Publ.

Bendikov, A. (1994). Asymptotic formulas for symmetric stable semigroups. *Exposition. Math. 12*(4), 381–384.

Benjamini, I., R. Pemantle, and Y. Peres (1995). Martin capacity for Markov chains. *Ann. Probab. 23*(3), 1332–1346.

Bergström, H. (1952). On some expansions of stable distribution functions. *Ark. Mat. 2*, 375–378.

Berman, S. M. (1983). Local nondeterminism and local times of general stochastic processes. *Ann. Inst. H. Poincaré Sect. B (N.S.) 19*(2), 189–207.

Bertoin, J. (1996). *Lévy Processes*. Cambridge: Cambridge University Press.

Bickel, P. J. and M. J. Wichura (1971). Convergence criteria for multiparameter stochastic processes and some applications. *Ann. Math. Statist. 42*, 1656–1670.

Billingsley, P. (1968). *Convergence of Probability Measures*. New York: John Wiley & Sons Inc.

Billingsley, P. (1995). *Probability and Measure* (Third ed.). New York: John Wiley & Sons Inc. A Wiley-Interscience Publication.

Blackwell, D. and L. Dubins (1962). Merging of opinions with increasing information. *Ann. Math. Statist. 33*, 882–886.

Blackwell, D. and L. E. Dubins (1975). On existence and non-existence of proper, regular, conditional distributions. *Ann. Probability 3*(5), 741–752.

Blumenthal, R. M. (1957). An extended Markov property. *Trans. Amer. Math. Soc. 85*, 52–72.

Blumenthal, R. M. and R. K. Getoor (1960a). A dimension theorem for sample functions of stable processes. *Illinois J. Math. 4*, 370–375.

Blumenthal, R. M. and R. K. Getoor (1960b). Some theorems on stable processes. *Trans. Amer. Math. Soc. 95*, 263–273.

Blumenthal, R. M. and R. K. Getoor (1962). The dimension of the set of zeros and the graph of a symmetric stable process. *Illinois J. Math. 6*, 308–316.

Blumenthal, R. M. and R. K. Getoor (1968). *Markov Processes and Potential Theory*. New York: Academic Press. Pure and Applied Mathematics, Vol. 29.

Bochner, S. (1955). *Harmonic Analysis and the Theory of Probability*. Berkeley and Los Angeles: University of California Press.

Borodin, A. N. (1986). On the character of convergence to Brownian local time. I, II. *Probab. Theory Relat. Fields 72*(2), 231–250, 251–277.

Borodin, A. N. (1988). On the weak convergence to Brownian local time. In *Probability theory and mathematical statistics (Kyoto, 1986)*, pp. 55–63. Berlin: Springer.

Burkholder, D. L. (1962). Successive conditional expectations of an integrable function. *Ann. Math. Statist. 33*, 887–893.

Burkholder, D. L. (1964). Maximal inequalities as necessary conditions for almost everywhere convergence. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 3*, 75–88 (1964).

Burkholder, D. L. (1973). Distribution function inequalities for martingales. *Ann. Probability 1*, 19–42.

Burkholder, D. L. (1975). One-sided maximal functions and H^p. *J. Functional Analysis 18*, 429–454.

Burkholder, D. L. and Y. S. Chow (1961). Iterates of conditional expectation operators. *Proc. Amer. Math. Soc. 12*, 490–495.

Burkholder, D. L., B. J. Davis, and R. F. Gundy (1972). Integral inequalities for convex functions of operators on martingales. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory*, pp. 223–240. Berkeley, Calif.: Univ. California Press.

Burkholder, D. L. and R. F. Gundy (1972). Distribution function inequalities for the area integral. *Studia Math. 44*, 527–544. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, VI.

Cabaña, E. (1991). The Markov property of the Brownian sheet associated with its wave components. In *Mathématiques appliquées aux sciences de l'ingénieur (Santiago, 1989)*, pp. 103–120. Toulouse: Cépaduès.

Cabaña, E. M. (1987). On two parameter Wiener process and the wave equation. *Acta Cient. Venezolana 38*(5-6), 550–555.

Cairoli, R. (1966). Produits de semi-groupes de transition et produits de processus. *Publ. Inst. Statist. Univ. Paris 15*, 311–384.

Cairoli, R. (1969). Un théorème de convergence pour martingales à indices multiples. *C. R. Acad. Sci. Paris Sér. A-B 269*, A587–A589.

Cairoli, R. (1970a). Décomposition de processus à indices doubles. *C. R. Acad. Sci. Paris Sér. A-B 270*, A669–A672.

Cairoli, R. (1970b). Processus croissant naturel associé à une classe de processus à indices doubles. *C. R. Acad. Sci. Paris Sér. A-B 270*, A1604–A1606.

Cairoli, R. (1970c). Une inégalité pour martingales à indices multiples et ses applications. In *Séminaire de Probabilités, IV (Univ. Strasbourg, 1968/1969)*, pp. 1–27. Lecture Notes in Mathematics, Vol. 124. Springer, Berlin.

Cairoli, R. (1971). Décomposition de processus à indices doubles. In *Séminaire de Probabilités, V (Univ. Strasbourg, année universitaire 1969-1970)*, pp. 37–57. Lecture Notes in Math., Vol. 191. Berlin: Springer.

Cairoli, R. (1979). Sur la convergence des martingales indexées par $\mathbf{n}\times\mathbf{n}$. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/1978)*, pp. 162–173. Berlin: Springer.

Cairoli, R. and R. C. Dalang (1996). *Sequential Stochastic Optimization*. New York: John Wiley & Sons Inc. A Wiley-Interscience Publication.

Cairoli, R. and J.-P. Gabriel (1979). Arrêt de certaines suites multiples de variables aléatoires indépendantes. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/1978)*, pp. 174–198. Berlin: Springer.

Cairoli, R. and J. B. Walsh (1975). Stochastic integrals in the plane. *Acta Math. 134*, 111–183.

Cairoli, R. and J. B. Walsh (1977a). Martingale representations and holomorphic processes. *Ann. Probability 5*(4), 511–521.

Cairoli, R. and J. B. Walsh (1977b). Prolongement de processus holomorphes. Cas “carré intégrable”. In *Séminaire de Probabilités, XI (Univ. Strasbourg, Strasbourg, 1975/1976)*, pp. 327–339. Lecture Notes in Math., Vol. 581. Berlin: Springer.

Cairoli, R. and J. B. Walsh (1977c). Some examples of holomorphic processes. In *Séminaire de Probabilités, XI (Univ. Strasbourg, Strasbourg, 1975/1976)*, pp. 340–348. Lecture Notes in Math., Vol. 581. Berlin: Springer.

Cairoli, R. and J. B. Walsh (1978). Régions d’arrêt, localisations et prolongements de martingales. *Z. Wahrsch. Verw. Gebiete 44*(4), 279–306.

Calderón, A. P. and A. Zygmund (1952). On the existence of certain singular integrals. *Acta Math. 88*, 85–139.

Carleson, L. (1958). On the connection between Hausdorff measures and capacity. *Ark. Mat. 3*, 403–406.

Carleson, L. (1983). *Selected Problems on Exceptional Sets*. Belmont, CA: Wadsworth. Selected reprints.

Čentsov, N. N. (1956). Wiener random fields depending on several parameters. *Dokl. Akad. Nauk. S.S.S.R. (NS) 106*, 607–609.

Chatterji, S. D. (1967). Comments on the martingale convergence theorem. In *Symposium on Probability Methods in Analysis (Loutraki, 1966)*, pp. 55–61. Berlin: Springer.

Chatterji, S. D. (1968). Martingale convergence and the Radon-Nikodym theorem in Banach spaces. *Math. Scand. 22*, 21–41.

Chen, Z. L. (1997). Properties of the polar sets of Brownian sheets. *J. Math. (Wuhan) 17*(3), 373–378.

Chow, Y. S. and H. Teicher (1997). *Probability theory* (Third ed.). New York: Springer-Verlag. Independence, Interchangeability, Martingales.

Chung, K. L. (1948). On the maximum partial sums of sequences of independent random variables. *Trans. Amer. Math. Soc. 64*, 205–233.

Chung, K. L. (1974). *A Course in Probability Theory* (Second ed.). Academic Press, New York-London. Probability and Mathematical Statistics, Vol. 21.

Chung, K. L. and P. Erdős (1952). On the application of the Borel-Cantelli lemma. *Trans. Amer. Math. Soc. 72*, 179–186.

Chung, K. L. and W. H. J. Fuchs (1951). On the distribution of values of sums of random variables. *Mem. Amer. Math. Soc. 1951*(6), 12.

Chung, K. L. and G. A. Hunt (1949). On the zeros of $\sum_1^n \pm 1$. *Ann. of Math. (2) 50*, 385–400.

Chung, K. L. and D. Ornstein (1962). On the recurrence of sums of random variables. *Bull. Amer. Math. Soc. 68*, 30–32.

Chung, K. L. and J. B. Walsh (1969). To reverse a Markov process. *Acta Math. 123*, 225–251.

Chung, K. L. and R. J. Williams (1990). *Introduction to Stochastic Integration* (Second ed.). Boston, MA: Birkhäuser Boston Inc.

Ciesielski, Z. (1959). On Haar functions and on the Schauder basis of the space $C_{\langle 0,\,1\rangle}$. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 7*, 227–232.

Ciesielski, Z. (1961). Hölder conditions for realizations of Gaussian processes. *Trans. Amer. Math. Soc. 99*, 403–413.

Ciesielski, Z. and J. Musielak (1959). On absolute convergence of Haar series. *Colloq. Math. 7*, 61–65.

Ciesielski, Z. and S. J. Taylor (1962). First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path. *Trans. Amer. Math. Soc. 103*, 434–450.

Csáki, E., M. Csörgő, A. Földes, and P. Révész (1989). Brownian local time approximated by a Wiener sheet. *Ann. Probab. 17*(2), 516–537.

Csáki, E., M. Csörgő, A. Földes, and P. Révész (1992). Strong approximation of additive functionals. *J. Theoret. Probab. 5*(4), 679–706.

Csáki, E., A. Földes, and Y. Kasahara (1988). Around Yor's theorem on the Brownian sheet and local time. *J. Math. Kyoto Univ. 28*(2), 373–381.

Csáki, E. and P. Révész (1983). Strong invariance for local times. *Z. Wahrsch. Verw. Gebiete 62*(2), 263–278.

Csörgő, M. and P. Révész (1978). How big are the increments of a multiparameter Wiener process? *Z. Wahrsch. Verw. Gebiete 42*(1), 1–12.

Csörgő, M. and P. Révész (1981). *Strong Approximations in Probability and Statistics*. New York: Academic Press Inc. [Harcourt Brace Jovanovich Publishers].

Csörgő, M. and P. Révész (1984). Three strong approximations of the local time of a Wiener process and their applications to invariance. In *Limit theorems in probability and statistics, Vol. I, II (Veszprém, 1982)*, pp. 223–254. Amsterdam: North-Holland.

Csörgő, M. and P. Révész (1985). On the stability of the local time of a symmetric random walk. *Acta Sci. Math. (Szeged) 48*(1-4), 85–96.

Csörgő, M. and P. Révész (1986). Mesure du voisinage and occupation density. *Probab. Theory Relat. Fields 73*(2), 211–226.

Dalang, R. C. and T. Mountford (1996). Nondifferentiability of curves on the Brownian sheet. *Ann. Probab. 24*(1), 182–195.

Dalang, R. C. and T. Mountford (1997). Points of increase of the Brownian sheet. *Probab. Theory Related Fields 108*(1), 1–27.

Dalang, R. C. and J. B. Walsh (1992). The sharp Markov property of the Brownian sheet and related processes. *Acta Math. 168*(3-4), 153–218.

Dalang, R. C. and J. B. Walsh (1993). Geography of the level sets of the Brownian sheet. *Probab. Theory Related Fields 96*(2), 153–176.

Dalang, R. C. and J. B. Walsh (1996). Local structure of level sets of the Brownian sheet. In *Stochastic analysis: random fields and measure-valued processes (Ramat Gan, 1993/1995)*, pp. 57–64. Ramat Gan: Bar-Ilan Univ.

Davis, B. and T. S. Salisbury (1988). Connecting Brownian paths. *Ann. Probab. 16*(4), 1428–1457.

de Acosta, A. (1983). A new proof of the Hartman–Wintner law of the iterated logarithm. *Ann. Probab. 11*(2), 270–276.

Dellacherie, C. and P.-A. Meyer (1978). *Probabilities and Potential*. Amsterdam: North-Holland Publishing Co.

Dellacherie, C. and P.-A. Meyer (1982). *Probabilities and Potential. B*. Amsterdam: North-Holland Publishing Co. Theory of martingales, Translated from the French by J. P. Wilson.

Dellacherie, C. and P.-A. Meyer (1988). *Probabilities and Potential. C*. Amsterdam: North-Holland Publishing Co. Potential theory for discrete and continuous semigroups, Translated from the French by J. Norris.

Donsker, M. D. (1952). Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems. *Ann. Math. Statistics 23*, 277–281.

Doob, J. L. (1962/1963). A ratio operator limit theorem. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 1*, 288–294.

Doob, J. L. (1984). *Classical Potential Theory and its Probabilistic Counterpart*. New York: Springer-Verlag.

Doob, J. L. (1990). *Stochastic Processes*. New York: John Wiley & Sons Inc. Reprint of the 1953 original, A Wiley-Interscience Publication.

Dorea, C. C. Y. (1982). A characterization of the multiparameter Wiener process and an application. *Proc. Amer. Math. Soc. 85*(2), 267–271.

Dorea, C. C. Y. (1983). A semigroup characterization of the multiparameter Wiener process. *Semigroup Forum 26*(3-4), 287–293.

Dozzi, M. (1988). On the local time of the multiparameter Wiener process and the asymptotic behaviour of an associated integral. *Stochastics 25*(3), 155–169.

Dozzi, M. (1989). *Stochastic Processes with a Multidimensional Parameter*. Harlow: Longman Scientific & Technical.

Dozzi, M. (1991). Two-parameter stochastic processes. In *Stochastic Processes and Related Topics (Georgenthal, 1990)*, pp. 17–43. Berlin: Akademie-Verlag.

Dubins, L. E. and J. Pitman (1980). A divergent, two-parameter, bounded martingale. *Proc. Amer. Math. Soc. 78*(3), 414–416.

Dudley, R. M. (1973). Sample functions of the Gaussian process. *Ann. Probability 1*(1), 66–103.

Dudley, R. M. (1984). A Course on Empirical Processes. In *École d'été de probabilités de Saint-Flour, XII—1982*, pp. 1–142. Berlin: Springer.

Dudley, R. M. (1989). *Real Analysis and Probability*. Pacific Grove, CA: Wadsworth & Brooks/Cole Advanced Books & Software.

Durrett, R. (1991). *Probability*. Pacific Grove, CA: Wadsworth & Brooks/Cole Advanced Books & Software. Theory and Examples.

Dvoretzky, A. and P. Erdős (1951). Some problems on random walk in space. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950.*, Berkeley and Los Angeles, pp. 353–367. University of California Press.

Dvoretzky, A., P. Erdős, and S. Kakutani (1950). Double points of paths of Brownian motion in n-space. *Acta Sci. Math. Szeged 12*(Leopoldo Fejer et Frederico Riesz LXX annos natis dedicatus, Pars B), 75–81.

Dvoretzky, A., P. Erdős, and S. Kakutani (1954). Multiple points of paths of Brownian motion in the plane. *Bull. Res. Council Israel 3*, 364–371.

Dvoretzky, A., P. Erdős, and S. Kakutani (1958). Points of multiplicity c of plane Brownian paths. *Bull. Res. Council Israel Sect. F 7F*, 175–180 (1958).

Dvoretzky, A., P. Erdős, S. Kakutani, and S. J. Taylor (1957). Triple points of Brownian paths in 3-space. *Proc. Cambridge Philos. Soc. 53*, 856–862.

Dynkin, E. B. (1965). *Markov Processes. Vols. I, II*. Publishers, New York: Academic Press Inc. Translated with the authorization and assistance of the author by J. Fabius, V. Greenberg, A. Maitra, G. Majone. Die Grundlehren der Mathematischen Wissenschaften, Bände 121, 122.

Dynkin, E. B. (1980). Markov processes and random fields. *Bull. Amer. Math. Soc. (N.S.) 3*(3), 975–999.

Dynkin, E. B. (1981a). Additive functionals of several time-reversible Markov processes. *J. Funct. Anal. 42*(1), 64–101.

Dynkin, E. B. (1981b). Harmonic functions associated with several Markov processes. *Adv. in Appl. Math. 2*(3), 260–283.

Dynkin, E. B. (1983). Markov processes as a tool in field theory. *J. Funct. Anal. 50*(2), 167–187.

Dynkin, E. B. (1984a). Local times and quantum fields. In *Seminar on stochastic processes, 1983 (Gainesville, Fla., 1983)*, pp. 69–83. Boston, Mass.: Birkhäuser Boston.

Dynkin, E. B. (1984b). Polynomials of the occupation field and related random fields. *J. Funct. Anal. 58*(1), 20–52.

Dynkin, E. B. (1985). Random fields associated with multiple points of the Brownian motion. *J. Funct. Anal. 62*(3), 397–434.

Dynkin, E. B. (1986). Generalized random fields related to self-intersections of the Brownian motion. *Proc. Nat. Acad. Sci. U.S.A. 83*(11), 3575–3576.

Dynkin, E. B. (1987). Self-intersection local times, occupation fields, and stochastic integrals. *Adv. in Math. 65*(3), 254–271.

Dynkin, E. B. (1988). Self-intersection gauge for random walks and for Brownian motion. *Ann. Probab. 16*(1), 1–57.

Dynkin, E. B. and R. J. Vanderbei (1983). Stochastic waves. *Trans. Amer. Math. Soc. 275*(2), 771–779.

Edgar, G. A. and L. Sucheston (1992). *Stopping Times and Directed Processes*. Cambridge: Cambridge University Press.

Ehm, W. (1981). Sample function properties of multiparameter stable processes. *Z. Wahrsch. Verw. Gebiete 56*(2), 195–228.

Eisenbaum, N. (1995). Une version sans conditionnement du théorème d'isomorphisms de Dynkin. In *Séminaire de Probabilités, XXIX*, pp. 266–289. Berlin: Springer.

Eisenbaum, N. (1997). Théorèmes limites pour les temps locaux d'un processus stable symétrique. In *Séminaire de Probabilités, XXXI*, pp. 216–224. Berlin: Springer.

Epstein, R. (1989). Some limit theorems for functionals of the Brownian sheet. *Ann. Probab. 17*(2), 538–558.

Erdős, P. (1942). On the law of the iterated logarithm. *Ann. Math. 43*, 419–436.

Erdős, P. and S. J. Taylor (1960a). Some intersection properties of random walk paths. *Acta Math. Acad. Sci. Hungar. 11*, 231–248.

Erdős, P. and S. J. Taylor (1960b). Some problems concerning the structure of random walk paths. *Acta Math. Acad. Sci. Hungar 11*, 137–162. (unbound insert).

Esquível, M. L. (1996). Points of rapid oscillation for the Brownian sheet via Fourier-Schauder series representation. In *Interaction between Functional Analysis, Harmonic Analysis, and Probability (Columbia, MO, 1994)*, pp. 153–162. New York: Dekker.

Etemadi, N. (1977). Collision problems of random walks in two-dimensional time. *J. Multivariate Anal. 7*(2), 249–264.

Etemadi, N. (1991). Maximal inequalities for partial sums of independent random vectors with multi-dimensional time parameters. *Comm. Statist. Theory Methods 20*(12), 3909–3923.

Ethier, S. N. (1998). An optional stopping theorem for nonadapted martingales. *Statist. Probab. Lett. 39*(3), 283–288.

Ethier, S. N. and T. G. Kurtz (1986). *Markov Processes. Characterization and Convergence*. New York: John Wiley & Sons Inc.

Evans, S. N. (1987a). Multiple points in the sample paths of a Lévy process. *Probab. Theory Related Fields 76*(3), 359–367.

Evans, S. N. (1987b). Potential theory for a family of several Markov processes. *Ann. Inst. H. Poincaré Probab. Statist. 23*(3), 499–530.

Feller, W. (1968). *An Introduction to Probability Theory and its Applications. Vol. I* (Third ed.). New York: John Wiley & Sons Inc.

Feller, W. (1968/1969). An extension of the law of the iterated logarithm to variables without variance. *J. Math. Mech. 18*, 343–355.

Feller, W. (1971). *An Introduction to Probability Theory and its Applications. Vol. II.* (Second ed.). New York: John Wiley & Sons Inc.

Feynman, R. J. (1948). Space-time approach to nonrelativistic quantuum mechanics. *Rev. Mod. Phys. 20*, 367–387.

Fitzsimmons, P. J. and B. Maisonneuve (1986). Excessive measures and Markov processes with random birth and death. *Probab. Theory Relat. Fields 72*(3), 319–336.

Fitzsimmons, P. J. and J. Pitman (1999). Kac's moment formula and the Feynman-Kac formula for additive functionals of a Markov process. *Stochastic Process. Appl. 79*(1), 117–134.

Fitzsimmons, P. J. and S. C. Port (1990). Local times, occupation times, and the Lebesgue measure of the range of a Lévy process. In *Seminar on Stochastic Processes, 1989 (San Diego, CA, 1989)*, pp. 59–73. Boston, MA: Birkhäuser Boston.

Fitzsimmons, P. J. and T. S. Salisbury (1989). Capacity and energy for multiparameter Markov processes. *Ann. Inst. H. Poincaré Probab. Statist. 25*(3), 325–350.

Föllmer, H. (1984a). Almost sure convergence of multiparameter martingales for Markov random fields. *Ann. Probab. 12*(1), 133–140.

Föllmer, H. (1984b). Von der Brownschen Bewegung zum Brownschen Blatt: einige neuere Richtungen in der Theorie der stochastischen Prozesse. In *Perspectives in mathematics*, pp. 159–190. Basel: Birkhäuser.

Fouque, J.-P., K. J. Hochberg, and E. Merzbach (Eds.) (1996). *Stochastic Analysis: Random Fields and Measure-Valued Processes*. Ramat Gan: Bar-Ilan University Gelbart Research Institute for Mathematical Sciences. Papers from the Binational France-Israel Symposium on the Brownian Sheet, held September 1993, and the Conference on Measure-valued Branching and Superprocesses, held May 1995, at Bar-Ilan University, Ramat Gan.

Frangos, N. E. and L. Sucheston (1986). On multiparameter ergodic and martingale theorems in infinite measure spaces. *Probab. Theory Relat. Fields 71*(4), 477–490.

Fukushima, M., Y. Ōshima, and M. Takeda (1994). *Dirichlet Forms and Symmetric Markov Processes*. Berlin: Walter de Gruyter & Co.

Gabriel, J.-P. (1977). Martingales with a countable filtering index set. *Ann. Probability 5*(6), 888–898.

Gänssler, P. (1983). *Empirical Processes*. Hayward, Calif.: Institute of Mathematical Statistics.

Gänssler, P. (1984). Limit theorems for empirical processes indexed by classes of sets allowing a finite-dimensional parametrization. *Probab. Math. Statist. 4*(1), 1–12.

Garsia, A. M. (1970). *Topics in Almost Everywhere Convergence*. Markham Publishing Co., Chicago, Ill. Lectures in Advanced Mathematics, 4.

Garsia, A. M. (1973). *Martingale Inequalities: Seminar Notes on Recent Progress*. W. A. Benjamin, Inc., Reading, Mass.-London-Amsterdam. Mathematics Lecture Notes Series.

Garsia, A. M., E. Rodemich, and H. Rumsey, Jr. (1970/1971). A real variable lemma and the continuity of paths of some Gaussian processes. *Indiana Univ. Math. J. 20*, 565–578.

Geman, D. and J. Horowitz (1980). Occupation densities. *Ann. Probab. 8*(1), 1–67.

Geman, D., J. Horowitz, and J. Rosen (1984). A local time analysis of intersections of Brownian paths in the plane. *Ann. Probab. 12*(1), 86–107.

Getoor, R. K. (1975). *Markov Processes: Ray Processes and Right Processes*. Berlin: Springer-Verlag. Lecture Notes in Mathematics, Vol. 440.

Getoor, R. K. (1979). Splitting times and shift functionals. *Z. Wahrsch. Verw. Gebiete 47*(1), 69–81.

Getoor, R. K. (1990). *Excessive Measures*. Boston, MA: Birkhäuser Boston Inc.

Getoor, R. K. and J. Glover (1984). Riesz decompositions in Markov process theory. *Trans. Amer. Math. Soc. 285*(1), 107–132.

Griffin, P. and J. Kuelbs (1991). Some extensions of the LIL via self-normalizations. *Ann. Probab. 19*(1), 380–395.

Gundy, R. F. (1969). On the class $L \log L$, martingales, and singular integrals. *Studia Math. 33*, 109–118.

Gut, A. (1978/1979). Moments of the maximum of normed partial sums of random variables with multidimensional indices. *Z. Wahrsch. Verw. Gebiete 46*(2), 205–220.

Hall, P. and C. C. Heyde (1980). *Martingale Limit Theory and its Application*. New York: Academic Press Inc. [Harcourt Brace Jovanovich Publishers]. Probability and Mathematical Statistics.

Hawkes, J. (1970/1971b). Some dimension theorems for the sample functions of stable processes. *Indiana Univ. Math. J. 20*, 733–738.

Hawkes, J. (1971a). On the Hausdorff dimension of the intersection of the range of a stable process with a Borel set. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 19*, 90–102.

Hawkes, J. (1976/1977). Intersections of Markov random sets. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 37*(3), 243–251.

Hawkes, J. (1978). Multiple points for symmetric Lévy processes. *Math. Proc. Cambridge Philos. Soc. 83*(1), 83–90.

Hawkes, J. (1979). Potential theory of Lévy processes. *Proc. London Math. Soc. (3) 38*(2), 335–352.

Helms, L. L. (1975). *Introduction to Potential Theory*. Robert E. Krieger Publishing Co., Huntington, N.Y. Reprint of the 1969 edition, Pure and Applied Mathematics, Vol. XXII.

Hendricks, W. J. (1972). Hausdorff dimension in a process with stable components—an interesting counterexample. *Ann. Math. Statist. 43*(2), 690–694.

Hendricks, W. J. (1973/1974). Multiple points for a process in R^2 with stable components. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 28*, 113–128.

Hendricks, W. J. (1979). Multiple points for transient symmetric Lévy processes in $\mathbf{R}^d$. *Z. Wahrsch. Verw. Gebiete 49*(1), 13–21.

Hewitt, E. and L. J. Savage (1955). Symmetric measures on Cartesian products. *Trans. Amer. Math. Soc. 80*, 470–501.

Hille, E. (1958). On roots and logarithms of elements of a complex Banach algebra. *Math. Ann. 136*, 46–57.

Hille, E. and R. S. Phillips (1957). *Functional Analysis and Semi-Groups*. Providence, R. I.: American Mathematical Society. rev. ed, American Mathematical Society Colloquium Publications, vol. 31.

Hirsch, F. (1995). Potential theory related to some multiparameter processes. *Potential Anal. 4*(3), 245–267.

Hirsch, F. and S. Song (1995a). Markov properties of multiparameter processes and capacities. *Probab. Theory Related Fields 103*(1), 45–71.

Hirsch, F. and S. Song (1995b). Symmetric Skorohod topology on n-variable functions and hierarchical Markov properties of n-parameter processes. *Probab. Theory Related Fields 103*(1), 25–43.

Hirsch, F. and S. Song (1995c). Une inégalité maximale pour certains processus de Markov à plusieurs paramètres. I. *C. R. Acad. Sci. Paris Sér. I Math. 320*(6), 719–722.

Hirsch, F. and S. Song (1995d). Une inégalité maximale pour certains processus de Markov à plusieurs paramètres. II. *C. R. Acad. Sci. Paris Sér. I Math. 320*(7), 867–870.

Hirsch, F. and S. Q. Song (1994). Propriétés de Markov des processus à plusieurs paramètres et capacités. *C. R. Acad. Sci. Paris Sér. I Math. 319*(5), 483–488.

Hoeffding, W. (1960). The strong law of large numbers for U-statistics. *University of North Carolina Institute of Statistics, Mimeo. Series*.

Hunt, G. A. (1956a). Markoff processes and potentials. *Proc. Nat. Acad. Sci. U.S.A. 42*, 414–418.

Hunt, G. A. (1956b). Semi-groups of measures on Lie groups. *Trans. Amer. Math. Soc. 81*, 264–293.

Hunt, G. A. (1957). Markoff processes and potentials. I, II. *Illinois J. Math. 1*, 44–93, 316–369.

Hunt, G. A. (1958). Markoff processes and potentials. III. *Illinois J. Math. 2*, 151–213.

Hunt, G. A. (1966). *Martingales et Processus de Markov*. Paris: Dunod. Monographies de la société Mathématique de France, No. 1.

Hürzeler, H. E. (1985). The optional sampling theorem for processes indexed by a partially ordered set. *Ann. Probab. 13*(4), 1224–1235.

Imkeller, P. (1984). Local times for a class of multiparameter processes. *Stochastics 12*(2), 143–157.

Imkeller, P. (1985). A stochastic calculus for continuous N-parameter strong martingales. *Stochastic Process. Appl. 20*(1), 1–40.

Imkeller, P. (1986). Local times of continuous N-parameter strong martingales. *J. Multivariate Anal. 19*(2), 348–365.

Imkeller, P. (1988). *Two-Parameter Martingales and Their Quadratic Variation*. Berlin: Springer-Verlag.

Itô, K. (1944). Stochastic integral. *Proc. Imp. Acad. Tokyo 20*, 519–524.

Itô, K. (1984). *Lectures on Stochastic Processes* (Second ed.). Distributed for the Tata Institute of Fundamental Research, Bombay. Notes by K. Muralidhara Rao.

Itô, K. and J. McKean, H. P. (1960). Potentials and the random walk. *Illinois J. Math. 4*, 119–132.

Itô, K. and J. McKean, Henry P. (1974). *Diffusion Processes and Their Sample Paths*. Berlin: Springer-Verlag. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125.

Ivanoff, G. and E. Merzbach (2000). *Set-Indexed Martingales*. Chapman & Hall/CRC, Boca Raton, FL.

Ivanova, B. G. and E. Mertsbakh (1992). Set-indexed stochastic processes and predictability. *Teor. Veroyatnost. i Primenen. 37*(1), 57–63.

Jacod, J. (1998). Rates of convergence to the local time of a diffusion. *Ann. Inst. H. Poincaré Probab. Statist. 34*(4), 505–544.

Janke, S. J. (1985). Recurrent sets for transient Lévy processes with bounded kernels. *Ann. Probab. 13*(4), 1204–1218.

Janson, S. (1997). *Gaussian Hilbert Spaces*. Cambridge: Cambridge University Press.

Kac, M. (1949). On deviations between theoretical and empirical distributions. *Proceedings of the National Academy of Sciences, U.S.A. 35*, 252–257.

Kac, M. (1951). On some connections between probability theory and differential and integral equations. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950*, Berkeley and Los Angeles, pp. 189–215. University of California Press.

Kahane, J.-P. (1982). Points multiples du mouvement brownien et des processus de Lévy symétriques, restreints à un ensemble compact de valeurs du temps. *C. R. Acad. Sci. Paris Sér. I Math. 295*(9), 531–534.

Kahane, J.-P. (1983). Points multiples des processus de Lévy symétriques stables restreints à un ensemble de valeurs du temps. In *Seminar on Harmonic Analysis, 1981–1982*, pp. 74–105. Orsay: Univ. Paris XI.

Kahane, J.-P. (1985). *Some Random Series of Functions* (Second ed.). Cambridge: Cambridge University Press.

Kakutani, S. (1944). Two-dimensional Brownian motion and harmonic functions. *Proc. Imp. Acad. Tokyo 20*, 706–714.

Kakutani, S. (1945). Markoff process and the Dirichlet problem. *Proc. Japan Acad. 21*, 227–233 (1949).

Kanda, M. (1982). Notes on polar sets for Lévy processes on the line. In *Functional analysis in Markov processes (Katata/Kyoto, 1981)*, pp. 227–234. Berlin: Springer.

Kanda, M. (1983). On the class of polar sets for a certain class of Lévy processes on the line. *J. Math. Soc. Japan 35*(2), 221–242.

Kanda, M. and M. Uehara (1981). On the class of polar sets for symmetric Lévy processes on the line. *Z. Wahrsch. Verw. Gebiete 58*(1), 55–67.

Karatsuba, A. A. (1993). *Basic Analytic Number Theory*. Berlin: Springer-Verlag. Translated from the second (1983) Russian edition and with a preface by Melvyn B. Nathanson.

Karatzas, I. and S. E. Shreve (1991). *Brownian Motion and Stochastic Calculus* (Second ed.). New York: Springer-Verlag.

Kargapolov, M. I. and J. I. Merzljakov (1979). *Fundamentals of the Theory of Groups*. New York: Springer-Verlag. Translated from the second Russian edition by Robert G. Burns.

Kellogg, O. D. (1967). *Foundations of Potential Theory*. Berlin: Springer-Verlag. Reprint from the first edition of 1929. Die Grundlehren der Mathematischen Wissenschaften, Band 31.

Kendall, W. S. (1980). Contours of Brownian processes with several-dimensional times. *Z. Wahrsch. Verw. Gebiete 52*(3), 267–276.

Kesten, H. and F. Spitzer (1979). A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. Verw. Gebiete 50*(1), 5–25.

Khoshnevisan, D. (1992). Level crossings of the empirical process. *Stochastic Process. Appl. 43*(2), 331–343.

Khoshnevisan, D. (1993). An embedding of compensated compound Poisson processes with applications to local times. *Ann. Probab. 21*(1), 340–361.

Khoshnevisan, D. (1994). A discrete fractal in $\mathbb{Z}_+^1$. *Proc. Amer. Math. Soc. 120*(2), 577–584.

Khoshnevisan, D. (1995). On the distribution of bubbles of the Brownian sheet. *Ann. Probab. 23*(2), 786–805.

Khoshnevisan, D. (1997a). Escape rates for Lévy processes. *Studia Sci. Math. Hungar. 33*(1-3), 177–183.

Khoshnevisan, D. (1997b). Some polar sets for the Brownian sheet. In *Séminaire de Probabilités, XXXI*, pp. 190–197. Berlin: Springer.

Khoshnevisan, D. (1998). On sums of iid random variables indexed by N parameters. In *Séminaire de Probabilités*. Berlin: Springer. (To appear).

Khoshnevisan, D. (1999). Brownian sheet images and Bessel-Riesz capacity. *Trans. Amer. Math. Soc. 351*(7), 2607–2622.

Khoshnevisan, D. and T. M. Lewis (1998). A law of the iterated logarithm for stable processes in random scenery. *Stochastic Process. Appl. 74*(1), 89–121.

Khoshnevisan, D. and Z. Shi (1998). Fast sets and points for fractional Brownian motion. *(Submitted for publication)*.

Khoshnevisan, D. and Z. Shi (1999). Brownian sheet and capacity. *Ann. Prob.*. (To appear).

Khoshnevisan, D. and Y. Xiao (1999). Level sets of additive Lévy processes. *(Submitted for publication)*.

Kinney, J. R. (1953). Continuity properties of sample functions of Markov processes. *Trans. Amer. Math. Soc. 74*, 280–302.

Kitagawa, T. (1951). Analysis of variance applied to function spaces. *Mem. Fac. Sci. Kyūsyū Univ. A. 6*, 41–53.

Knight, F. B. (1981). *Essentials of Brownian motion and Diffusion*. Providence, R.I.: American Mathematical Society.

Kochen, S. and C. Stone (1964). A note on the Borel-Cantelli lemma. *Illinois J. Math. 8*, 248–251.

Kolmogorov, A. N. and V. M. Tihomirov (1961). ε-entropy and ε-capacity of sets in functional space. *Amer. Math. Soc. Transl. (2) 17*, 277–364.

Körezlioğlu, H., G. Mazziotto, and J. Szpirglas (Eds.) (1981). *Processus aléatoires à deux indices*. Berlin: Springer. Papers from the E.N.S.T.-C.N.E.T. Colloquium held in Paris, June 30–July 1, 1980.

Krengel, U. and R. Pyke (1987). Uniform pointwise ergodic theorems for classes of averaging sets and multiparameter subadditive processes. *Stochastic Process. Appl. 26*(2), 289–296.

Krickeberg, K. (1963). *Wahrscheinlichkeitstheorie*. B. G. Teubner Verlagsgesellschaft, Stuttgart.

Krickeberg, K. (1965). *Probability Theory*. Addison-Wesley Publishing Co., Inc., Reading, Mass.-London.

Krickeberg, K. and C. Pauc (1963). Martingales et dérivation. *Bull. Soc. Math. France 91*, 455–543.

Kunita, H. and S. Watanabe (1967). On square integrable martingales. *Nagoya Math. J. 30*, 209–245.

Kuroda, K. and H. Manaka (1987). The interface of the Ising model and the Brownian sheet. In *Proceedings of the symposium on statistical mechanics of phase transitions—mathematical and physical aspects (Trebon, 1986)*, Volume 47, pp. 979–984.

Kuroda, K. and H. Manaka (1998). Limit theorem related to an interface of three-dimensional Ising model. *Kobe J. Math. 15*(1), 17–39.

Kuroda, K. and H. Tanemura (1988). Interacting particle system and Brownian sheet. *Keio Sci. Tech. Rep. 41*(1), 1–16.

Kwon, J. S. (1994). The law of large numbers for product partial sum processes indexed by sets. *J. Multivariate Anal. 49*(1), 76–86.

Lacey, M. T. (1990). Limit laws for local times of the Brownian sheet. *Probab. Theory Related Fields 86*(1), 63–85.

Lachout, P. (1988). Billingsley-type tightness criteria for multiparameter stochastic processes. *Kybernetika (Prague) 24*(5), 363–371.

Lamb, C. W. (1973). A short proof of the martingale convergence theorem. *Proc. Amer. Math. Soc. 38*, 215–217.

Landkof, N. S. (1972). *Foundations of Modern Potential Theory*. New York: Springer-Verlag. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

Lawler, G. F. (1991). *Intersections of Random Walks*. Boston, MA: Birkhäuser Boston Inc.

Le Gall, J.-F. (1987). Temps locaux d'intersection et points multiples des processus de Lévy. In *Séminaire de Probabilités, XXI*, pp. 341–374. Berlin: Springer.

Le Gall, J.-F. (1992). Some Properties of Planar Brownian Motion. In *École d'Été de Probabilités de Saint-Flour XX—1990*, pp. 111–235. Berlin: Springer.

Le Gall, J.-F. and J. Rosen (1991). The range of stable random walks. *Ann. Probab. 19*(2), 650–705.

Le Gall, J.-F., J. S. Rosen, and N.-R. Shieh (1989). Multiple points of Lévy processes. *Ann. Probab. 17*(2), 503–515.

LeCam, L. (1957). Convergence in distribution of stochastic processes. *University of California Pub. Stat. 2*(2).

Ledoux, M. (1981). Classe $L\log L$ et martingales fortes à paramètre bidimensionnel. *Ann. Inst. H. Poincaré Sect. B (N.S.) 17*(3), 275–280.

Ledoux, M. (1996). *Lectures on Probability Theory and Statistics*. Berlin: Springer-Verlag. Lectures from the 24th Saint-Flour Summer School held July 7–23, 1994, Edited by P. Bernard (with Dobrushin, R. and Groeneboom, P.).

Ledoux, M. and M. Talagrand (1991). *Probability in Banach Spaces*. Berlin: Springer-Verlag.

Lévy, P. (1965). *Processus stochastiques et mouvement brownien*. Gauthier-Villars & Cie, Paris. Suivi d'une note de M. Loève. Deuxième édition revue et augmentée.

Lyons, R. (1990). Random walks and percolation on trees. *Ann. Probab. 18*(3), 931–958.

Madras, N. and G. Slade (1993). *The Self-Avoiding Walk*. Boston, MA: Birkhäuser Boston Inc.

Mandelbaum, A. and R. J. Vanderbei (1981). Optimal stopping and supermartingales over partially ordered sets. *Z. Wahrsch. Verw. Gebiete 57*(2), 253–264.

Mandelbrot, B. B. (1982). *The Fractal Geometry of Nature*. San Francisco, Calif.: W. H. Freeman and Co. Schriftenreihe für den Referenten. [Series for the Referee].

Marcus, M. B. and J. Rosen (1992). Sample path properties of the local times of strongly symmetric Markov processes via Gaussian processes. *Ann. Probab. 20*(4), 1603–1684.

Mattila, P. (1995). *Geometry of Sets and Measures in Euclidean Spaces*. Cambridge: Cambridge University Press.

Mazziotto, G. (1988). Two-parameter Hunt processes and a potential theory. *Ann. Probab. 16*(2), 600–619.

Mazziotto, G. and E. Merzbach (1985). Regularity and decomposition of two-parameter supermartingales. *J. Multivariate Anal. 17*(1), 38–55.

Mazziotto, G. and J. Szpirglas (1981). Un exemple de processus à deux indices sans l'hypothèse F4. In *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, pp. 673–688. Berlin: Springer.

Mazziotto, G. and J. Szpirglas (1982). Optimal stopping for two-parameter processes. In *Advances in filtering and optimal stochastic control (Cocoyoc, 1982)*, pp. 239–245. Berlin: Springer.

Mazziotto, G. and J. Szpirglas (1983). Arrêt optimal sur le plan. *Z. Wahrsch. Verw. Gebiete 62*(2), 215–233.

McKean, Henry P., J. (1955a). Hausdorff-Besicovitch dimension of Brownian motion paths. *Duke Math. J. 22*, 229–234.

McKean, Henry P., J. (1955b). Sample functions of stable processes. *Ann. of Math. (2) 61*, 564–579.

Merzbach, E. and D. Nualart (1985). Different kinds of two-parameter martingales. *Israel J. Math. 52*(3), 193–208.

Millar, P. W. (1978). A path decomposition for Markov processes. *Ann. Probability 6*(2), 345–348.

Millet, A. and L. Sucheston (1980). Convergence et régularité des martingales à indices multiples. *C. R. Acad. Sci. Paris Sér. A-B 291*(2), A147–A150.

Mountford, T. S. (1993). Estimates of the Hausdorff dimension of the boundary of positive Brownian sheet components. In *Séminaire de Probabilités, XXVII*, pp. 233–255. Berlin: Springer.

Mountford, T. S. (1999). Brownian bubbles and the local time of the brownian sheet. (In progress).

Munkres, J. R. (1975). *Topology: A First Course*. Englewood Cliffs, N.J.: Prentice-Hall Inc.

Nagasawa, M. (1964). Time reversions of Markov processes. *Nagoya Math. J. 24*, 177–204.

Neveu, J. (1975). *Discrete-Parameter Martingales* (Revised ed.). Amsterdam: North-Holland Publishing Co. Translated from the French by T. P. Speed, North-Holland Mathematical Library, Vol. 10.

Nualart, D. (1985). Variations quadratiques et inégalités pour les martingales à deux indices. *Stochastics 15*(1), 51–63.

Nualart, D. (1995). *The Malliavin Calculus and Related Topics*. New York: Springer-Verlag.

Nualart, E. (2001a). Ph. d. thesis (in preparation).

Nualart, E. (2001b). Potential theory for hyperbolic spde's. *Preprint*.

Orey, S. (1967). Polar sets for processes with stationary independent increments. In *Markov Processes and Potential Theory*, pp. 117–126. New York: Wiley. (Proc. Sympos. Math. Res. Center, Madison, Wis., 1967).

Orey, S. and W. E. Pruitt (1973). Sample functions of the N-parameter Wiener process. *Ann. Probability 1*(1), 138–163.

Ornstein, D. S. (1969). Random walks. I, II. *Trans. Amer. Math. Soc. 138 (1969), 1-43; ibid. 138*, 45–60.

Oxtoby, J. C. and S. Ulam (1939). On the existence of a measure invariant under a transformation. *Ann. Math. 40*(2), 560–566.

Paley, R. E. A. C. and A. Zygmund (1932). A note on analytic functions in the unit circle. *Proc. Camb. Phil. Soc. 28*, 366–272.

Paranjape, S. R. and C. Park (1973a). Distribution of the supremum of the two-parameter Yeh-Wiener process on the boundary. *J. Appl. Probability 10*, 875–880.

Paranjape, S. R. and C. Park (1973b). Laws of iterated logarithm of multiparameter Wiener processes. *J. Multivariate Anal. 3*, 132–136.

Park, C. and D. L. Skoug (1978). Distribution estimates of barrier-crossing probabilities of the Yeh-Wiener process. *Pacific J. Math. 78*(2), 455–466.

Park, W. J. (1975). The law of the iterated logarithm for Brownian sheets. *J. Appl. Probability 12*(4), 840–844.

Pemantle, R., Y. Peres, and J. W. Shapiro (1996). The trace of spatial Brownian motion is capacity-equivalent to the unit square. *Probab. Theory Related Fields 106*(3), 379–399.

Peres, Y. (1996a). Intersection-equivalence of Brownian paths and certain branching processes. *Comm. Math. Phys. 177*(2), 417–434.

Peres, Y. (1996b). Remarks on intersection-equivalence and capacity-equivalence. *Ann. Inst. H. Poincaré Phys. Théor. 64*(3), 339–347.

Perkins, E. (1982). Weak invariance principles for local time. *Z. Wahrsch. Verw. Gebiete 60*(4), 437–451.

Pollard, D. (1984). *Convergence of Stochastic Processes*. New York: Springer-Verlag.

Pólya, G. (1921). Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend der Irrfahrt im Straßennetz. *Math. Annalen 84*, 149–160.

Pólya, G. (1923). On the zeros of an integral function represented by Fourier's intergal. *Mess. of Math. 52*, 185–188.

Port, S. C. (1988). Occupation time and the Lebesgue measure of the range for a Lévy process. *Proc. Amer. Math. Soc. 103*(4), 1241–1248.

Port, S. C. and C. J. Stone (1971a). Infinitely divisible processes and their potential theory, I. *Ann. Inst. Fourier (Grenoble) 21*(2), 157–275.

Port, S. C. and C. J. Stone (1971b). Infinitely divisible processes and their potential theory, II. *Ann. Inst. Fourier (Grenoble) 21*(4), 179–265.

Pruitt, W. E. (1969/1970). The Hausdorff dimension of the range of a process with stationary independent increments. *J. Math. Mech. 19*, 371–378.

Pruitt, W. E. (1975). Some dimension results for processes with independent increments. In *Stochastic processes and related topics (Proc. Summer Res. Inst. on Statist. Inference for Stochastic Processes, Indiana Univ., Bloomington, Ind., 1974, Vol. 1; dedicated to Jerzy Neyman)*, New York, pp. 133–165. Academic Press.

Pruitt, W. E. and S. J. Taylor (1969). The potential kernel and hitting probabilities for the general stable process in $\mathbf{R}^{\mathbf{N}}$. *Trans. Amer. Math. Soc. 146*, 299–321.

Pyke, R. (1973). Partial sums of matrix arrays, and Brownian sheets. In *Stochastic analysis (a tribute to the memory of Rollo Davidson)*, pp. 331–348. London: Wiley.

Pyke, R. (1983). A uniform central limit theorem for partial-sum processes indexed by sets. In *Probability, statistics and analysis*, pp. 219–240. Cambridge: Cambridge Univ. Press.

Pyke, R. (1985). Opportunities for set-indexed empirical and quantile processes in inference. In *Proceedings of the 45th session of the International Statistical Institute, Vol. 4 (Amsterdam, 1985)*, Volume 51, pp. No. 25.2, 11.

Rao, M. (1987). On polar sets for Lévy processes. *J. London Math. Soc. (2) 35*(3), 569–576.

Ren, J. G. (1990). Topologie p-fine sur l'espace de Wiener et théorème des fonctions implicites. *Bull. Sci. Math. (2) 114*(2), 99–114.

Révész, P. (1981). A strong invariance principle of the local time of RVs with continuous distribution. *Studia Sci. Math. Hungar. 16*(1-2), 219–228.

Révész, P. (1990). *Random Walk in Random and Nonrandom Environments*. Teaneck, NJ: World Scientific Publishing Co. Inc.

Revuz, D. (1984). *Markov Chains* (Second ed.). Amsterdam: North-Holland Publishing Co.

Revuz, D. and M. Yor (1994). *Continuous Martingales and Brownian Motion* (Second ed.). Berlin: Springer-Verlag.

Ricci, F. and E. M. Stein (1992). Multiparameter singular integrals and maximal functions. *Ann. Inst. Fourier (Grenoble) 42*(3), 637–670.

Riesz, F. and B. Sz.-Nagy (1955). *Functional Analysis*. New York: Fredrick Ungar Publishing Company. Seventh Printing, 1978. Translated from the second French edition by Leo F. Boron.

Rogers, C. A. and S. J. Taylor (1962). On the law of the iterated logarithm. *J. London Math. Soc. 37*, 145–151.

Rogers, L. C. G. (1989). Multiple points of Markov processes in a complete metric space. In *Séminaire de Probabilités, XXIII*, pp. 186–197. Berlin: Springer.

Rogers, L. C. G. and D. Williams (1987). *Diffusions, Markov Processes, and Martingales. Vol. 2*. New York: John Wiley & Sons Inc. Itô calculus.

Rogers, L. C. G. and D. Williams (1994). *Diffusions, Markov Processes, and Martingales. Vol. 1* (Second ed.). Chichester: John Wiley & Sons Ltd. Foundations.

Rosen, J. (1983). A local time approach to the self-intersections of Brownian paths in space. *Comm. Math. Phys. 88*(3), 327–338.

Rosen, J. (1984). Self-intersections of random fields. *Ann. Probab. 12*(1), 108–119.

Rosen, J. (1991). Second order limit laws for the local times of stable processes. In *Séminaire de Probabilités, XXV*, pp. 407–424. Berlin: Springer.

Rosen, J. (1993). Uniform invariance principles for intersection local times. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, pp. 241–247. Boston, MA: Birkhäuser Boston.

Rota, G.-C. (1962). An "Alternierende Verfahren" for general positive operators. *Bull. Amer. Math. Soc. 68*, 95–102.

Rota, G.-C. (1969). Syposium on Ergodic Theory, Tulane University. Presented in October 1969.

Rota, G.-C. (1997). *Indiscrete Thoughts*. Boston, MA: Birkhäuser Boston Inc. With forewords by Reuben Hersh and Robert Sokolowski, Edited with notes and an epilogue by Fabrizio Palombi.

Royden, H. L. (1968). *Real Analysis* (Second ed.). New York: Macmillan Publishing Company.

Rozanov, Yu. A. (1982). *Markov Random Fields*. New York: Springer-Verlag. Translated from the Russian by Constance M. Elson.

Rudin, W. (1973). *Functional Analysis* (First ed.). New York: McGraw-Hill Inc.

Rudin, W. (1974). *Real and complex analysis* (Third ed.). New York: McGraw-Hill Book Co.

Salisbury, T. S. (1988). Brownian bitransforms. In *Seminar on Stochastic Processes, 1987 (Princeton, NJ, 1987)*, pp. 249–263. Boston, MA: Birkhäuser Boston.

Salisbury, T. S. (1992). A low intensity maximum principle for bi-Brownian motion. *Illinois J. Math. 36*(1), 1–14.

Salisbury, T. S. (1996). Energy, and intersections of Markov chains. In *Random discrete structures (Minneapolis, MN, 1993)*, pp. 213–225. New York: Springer.

Sato, K. (1999). *Lévy Processes and Infinitely Divisble Processes*. Cambridge: Cambridge University Press.

Segal, I. E. (1954). Abstract probability spaces and a theorem of Kolmogoroff. *Amer. J. Math. 76*, 721–732.

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*. New York: John Wiley & Sons Inc. Wiley Series in Probability and Mathematical Statistics.

Sharpe, M. (1988). *General Theory of Markov Processes*. Boston, MA: Academic Press Inc.

Shieh, N. R. (1982). Strong differentiation and martingales in product spaces. *Math. Rep. Toyama Univ. 5*, 29–36.

Shieh, N.-R. (1989). Collisions of Lévy processes. *Proc. Amer. Math. Soc. 106*(2), 503–506.

Shorack, G. R. and R. T. Smythe (1976). Inequalities for $\max \mid S_{\mathbf{k}} \mid / b_{\mathbf{k}}$ where $\mathbf{k} \in \mathbf{N}^r$. *Proc. Amer. Math. Soc. 54*, 331–336.

Slepian, D. (1962). The one-sided barrier problem for Gaussian noise. *Bell System Tech. J. 41*, 463–501.

Smythe, R. T. (1973). Strong laws of large numbers for r-dimensional arrays of random variables. *Ann. Probability 1*(1), 164–170.

Smythe, R. T. (1974a). Convergence de sommes de variables aléatoires indicées par des ensembles partiellement ordonnés. *Ann. Sci. Univ. Clermont 51*(9), 43–46. Colloque Consacré au Calcul des Probabilités (Univ. Clermont, Clermont-Ferrand, 1973).

Smythe, R. T. (1974b). Sums of independent random variables on partially ordered sets. *Ann. Probability 2*, 906–917.

Song, R. G. (1988). Optimal stopping for general stochastic processes indexed by a lattice-ordered set. *Acta Math. Sci. (English Ed.) 8*(3), 293–306.

Spitzer, F. (1958). Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc. 87*, 187–197.

Spitzer, F. (1964). *Principles of Random Walk*. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London. The University Series in Higher Mathematics.

Stein, E. M. (1961). On the maximal ergodic theorem. *Proc. Nat. Acad. Sci. U.S.A. 47*, 1894–1897.

Stein, E. M. (1970). *Singular Integrals and Differentiability Properties of Functions*. Princeton, N.J.: Princeton University Press. Princeton Mathematical Series, No. 30.

Stein, E. M. (1993). *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton, NJ: Princeton University Press. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

Stein, E. M. and G. Weiss (1971). *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton, N.J.: Princeton University Press. Princeton Mathematical Series, No. 32.

Stoll, A. (1987). Self-repellent random walks and polymer measures in two dimensions. In *Stochastic processes—mathematics and physics, II (Bielefeld, 1985)*, pp. 298–318. Berlin: Springer.

Stoll, A. (1989). Invariance principles for Brownian intersection local time and polymer measures. *Math. Scand. 64*(1), 133–160.

Stone, C. J. (1969). On the potential operator for one-dimensional recurrent random walks. *Trans. Amer. Math. Soc. 136*, 413–426.

Stout, W. F. (1974). *Almost Sure Convergence*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London. Probability and Mathematical Statistics, Vol. 24.

Strassen, V. (1965/1966). A converse to the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 4*, 265–268.

Stratonovich, R. L. (1966). A new representation for stochastic integrals and equations. *SIAM J. of Control 4*, 362–371.

Stroock, D. W. (1993). *Probability Theory, an Analytic View*. Cambridge: Cambridge University Press.

Sucheston, L. (1983). On one-parameter proofs of almost sure convergence of multiparameter processes. *Z. Wahrsch. Verw. Gebiete 63*(1), 43–49.

Takeuchi, J. (1964a). A local asymptotic law for the transient stable process. *Proc. Japan Acad. 40*, 141–144.

Takeuchi, J. (1964b). On the sample paths of the symmetric stable processes in spaces. *J. Math. Soc. Japan 16*, 109–127.

Takeuchi, J. and S. Watanabe (1964). Spitzer's test for the Cauchy process on the line. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 3*, 204–210 (1964).

Taylor, S. J. (1953). The Hausdorff α-dimensional measure of Brownian paths in n-space. *Proc. Cambridge Philos. Soc. 49*, 31–39.

Taylor, S. J. (1955). The α-dimensional measure of the graph and set of zeros of a Brownian path. *Proc. Cambridge Philos. Soc. 51*, 265–274.

Taylor, S. J. (1961). On the connexion between Hausdorff measures and generalized capacity. *Proc. Cambridge Philos. Soc. 57*, 524–531.

Taylor, S. J. (1966). Multiple points for the sample paths of the symmetric stable process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 5*, 247–264.

Taylor, S. J. (1973). Sample path properties of processes with stationary independent increments. In *Stochastic analysis (a tribute to the memory of Rollo Davidson)*, pp. 387–414. London: Wiley.

Taylor, S. J. (1986). The measure theory of random fractals. *Math. Proc. Cambridge Philos. Soc. 100*(3), 383–406.

Testard, F. (1985). Quelques propriétés géométriques de certains processus gaussiens. *C. R. Acad. Sci. Paris Sér. I Math. 300*(14), 497–500.

Testard, F. (1986). Dimension asymétrique et ensembles doublement non polaires. *C. R. Acad. Sci. Paris Sér. I Math. 303*(12), 579–581.

Tihomirov, V. M. (1963). The works of A. N. Kolmogorov on ε-entropy of function classes and superpositions of functions. *Uspehi Mat. Nauk 18*(5 (113)), 55–92.

Vanderbei, R. J. (1983). Toward a stochastic calculus for several Markov processes. *Adv. in Appl. Math. 4*(2), 125–144.

Vanderbei, R. J. (1984). Probabilistic solution of the Dirichlet problem for biharmonic functions in discrete space. *Ann. Probab. 12*(2), 311–324.

Vares, M. E. (1983). Local times for two-parameter Lévy processes. *Stochastic Process. Appl. 15*(1), 59–82.

Walsh, J. B. (1972). Transition functions of Markov processes. In *Séminaire de Probabilités, VI (Univ. Strasbourg, année universitaire 1970–1971)*, pp. 215–232. Lecture Notes in Math., Vol. 258. Berlin: Springer.

Walsh, J. B. (1978/1979). Convergence and regularity of multiparameter strong martingales. *Z. Wahrsch. Verw. Gebiete 46*(2), 177–192.

Walsh, J. B. (1981). Optional increasing paths. In *Two-Index Random Processes (Paris, 1980)*, pp. 172–201. Berlin: Springer.

Walsh, J. B. (1982). Propagation of singularities in the Brownian sheet. *Ann. Probab. 10*(2), 279–288.

Walsh, J. B. (1986a). An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, pp. 265–439. Berlin: Springer.

Walsh, J. B. (1986b). Martingales with a multidimensional parameter and stochastic integrals in the plane. In *Lectures in probability and statistics (Santiago de Chile, 1986)*, pp. 329–491. Berlin: Springer.

Watson, G. N. (1995). *A Treatise on the Theory of Bessel Functions*. Cambridge: Cambridge University Press. Reprint of the second (1944) edition.

Weber, M. (1983). Polar sets of some Gaussian processes. In *Probability in Banach Spaces, IV (Oberwolfach, 1982)*, pp. 204–214. Berlin: Springer.

Weinryb, S. (1986). Étude asymptotique par des mesures de $\mathbf{r}^3$ de saucisses de Wiener localisées. *Probab. Theory Relat. Fields 73*(1), 135–148.

Weinryb, S. and M. Yor (1988). Le mouvement brownien de Lévy indexé par $\mathbf{R}^3$ comme limite centrale de temps locaux d'intersection. In *Séminaire de Probabilités, XXII*, pp. 225–248. Berlin: Springer.

Weinryb, S. and M. Yor (1993). Théorème central limite pour l'intersection de deux saucisses de Wiener indépendantes. *Probab. Theory Related Fields 97*(3), 383–401.

Wermer, J. (1981). *Potential Theory* (Second ed.). Berlin: Springer.

Werner, W. (1993). Sur les singularités des temps locaux d'intersection du mouvement brownien plan. *Ann. Inst. H. Poincaré Probab. Statist. 29*(3), 391–418.

Wichura, M. J. (1973). Some Strassen-type laws of the iterated logarithm for multiparameter stochastic processes with independent increments. *Ann. Probab. 1*, 272–296.

Williams, D. (1970). Decomposing the Brownian path. *Bull. Amer. Math. Soc. 76*, 871–873.

Williams, D. (1974). Path decomposition and continuity of local time for one-dimensional diffusions. I. *Proc. London Math. Soc. (3) 28*, 738–768.

Wong, E. (1989). Multiparameter martingale and Markov process. In *Stochastic Differential Systems (Bad Honnef, 1988)*, pp. 329–336. Berlin: Springer.

Yor, M. (1983). Le drap brownien comme limite en loi de temps locaux linéaires. In *Seminar on probability, XVII*, pp. 89–105. Berlin: Springer.

Yor, M. (1992). *Some Aspects of Brownian Motion. Part I*. Basel: Birkhäuser Verlag. Some special functionals.

Yor, M. (1997). *Some Aspects of Brownian Motion. Part II*. Basel: Birkhäuser Verlag. Some recent martingale problems.

Yosida, K. (1958). On the differentiability of semigroups of linear operators. *Proc. Japan Acad. 34*, 337–340.

Yosida, K. (1995). *Functional Analysis*. Berlin: Springer-Verlag. Reprint of the sixth (1980) edition.

Zhang, R. C. (1985). Markov properties of the generalized Brownian sheet and extended OUP_2. *Sci. Sinica Ser. A 28*(8), 814–825.

Zinčenko, N. M. (1975). On the probability of exit of the Wiener random field from some surface. *Teor. Verojatnost. i Mat. Statist.* (Vyp. 13), 62–69, 162.

Zygmund, A. (1988). *Trigonometric Series. Vol. I, II*. Cambridge: Cambridge University Press. Reprint of the 1979 edition.