Five Lectures on Brownian Sheet:
Summer Internship Program
University of Wisconsin–Madison

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Preface

These notes are based on five 1-hour lectures on Brownian sheet and potential theory, given at the *Center for Mathematical Sciences* at the *University of Wisconsin-Madison*, July 2001. While the notes cover the material in more depth, and while they contain more details, I have tried to remain true to the basic outline of the lectures. A more detailed set of notes on potential theory, see my *EPFL* notes, although the material of these lectures covers other subjects, as well. My *EPFL* notes are publicly available at HTTP://WWW.MATH.UTAH.EDU/˜DAVAR/LECTURE-NOTES.HTML. Finally, a much more complete theory can be found in my forthcoming book *Multiparameter Processes: An Introduction to Random Fields*, to be published by *Springer–Verlag Monographs in Mathematics*. All references to "*MPP*" are to this book.

I am grateful to Professor J. Kuelbs for his invitation, and to all the participants for the interest and support. I also wish to thank the *National Science Foundation* for their continued support, both of my work, as well as the wonderful Summer Internship Program at the *University of Wisconsin-Madison*.

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Lecture 1

Čentsov's Representation

A one-dimensional Brownian sheet is a 2-parameter[1], centered Gaussian process $B = \{B(s,t);\, s,t \geq 0\}$ whose covariance is given by

$$\mathbb{E}\{B(s,t)B(s',t')\} = \min(s,s') \times \min(t,t'), \qquad \forall s,s',t,t' \geq 0.$$

There are many ways to arrive at such a process; one of the quickest is via fluctuation theory for 2-parameter random walks. Imagine a sequence $\{\xi_{i,j};\, i,j \geq 1\}$ of i.i.d. random variables that take their values in $\{0,1\}$ with $\mathbb{P}\{\xi_{1,1} = 0\} = \mathbb{P}\{\xi_{1,1} = 1\} = \frac{1}{2}$. For instance, think of each "*site*" (i,j) as "*infected*" if $\xi_{i,j} = 1$, otherwise $\xi_{i,j} = 0$. Then, the number of infected sites in a large box $[0,n] \times [0,m]$ is

$$S(n,m) = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq m} \xi_{i,j}, \qquad \forall n,m \geq 1. \tag{0.1}$$

A natural way to normalize this quantity is to set

$$\xi'_{i,j} = \frac{\xi_{i,j} - \mathbb{E}\{\xi_{i,j}\}}{\sqrt{\mathrm{Var}\{\xi_{i,j}\}}} = 2\xi_{i,j} - 1, \qquad S'(n,m) = \frac{S(n,m) - \mathbb{E}\{S(n,m)\}}{\sqrt{\mathrm{Var}\{S(n,m)\}}} = \frac{2S(n,m) - nm}{\sqrt{nm}}.$$

Note that $S(n,m)$ is a sum of nm i.i.d. variables, each with mean $\frac{1}{2}$ and variance $\frac{1}{4}$, and $S'(n,m)$ is its standardization. Thus, we can replace $\xi_{i,j}$ by $\xi'_{i,j}$ everywhere, and assume that Eq. (0.1) holds, where $\mathbb{E}\{\xi_{i,j}\} = 0$ and $\mathbb{E}\{\xi_{i,j}^2\} = 1$. With this sudden change of notation in mind, $S(n,m)$ is a sum of nm i.i.d. random variables with mean 0 and variance 1. Thus, by the central limit theorem of De Moivre and Laplace, $(nm)^{-\frac{1}{2}} S(n,m)$ converges in distribution to a standard normal law, as $n,m \to \infty$. Of particular importance is the case where n and m go to infinity at the same rate. That is, when $n = \lfloor Ns \rfloor$ and $m = \lfloor Nt \rfloor$, where $s,t \in [0,1]$ are fixed, but $N \to \infty$. In this case, $N^{-1} S(\lfloor Ns \rfloor, \lfloor Nt \rfloor)$ converges to

[1] Much of what we do here can be extended to more parameters, but such extensions are not of central importance to these lectures.

a mean zero normal law with variance $\sqrt{ij}$. Note that this is the law of $B(i,j)$, and one can show, in fact, that in a suitable sense, the entire random function $\{N^{-1}S(\lfloor Ns \rfloor, \lfloor Nt \rfloor);\ (s,t) \in [0,1]^2\}$ converges weakly to the random function $\{B(s,t);\ (s,t) \in [0,1]^2\}$. To convince yourself, check, for instance, that the covariance between $N^{-1}S(\lfloor Ns \rfloor, \lfloor Nt \rfloor)$ and $N^{-1}S(\lfloor Ns' \rfloor, \lfloor Nt' \rfloor)$ converges, as $N \to \infty$, to that between $B(s,t)$ and $B(s',t')$, which is $\min(s,s') \times \min(t,t')$.

In the above model, one can relax the independence assumption to something involving "asymptotic independence" (e.g., strong mixing, etc.) to obtain more realistic models.

1 White Noise

Recall that a column vector $\mathbf{X} = (X_1, \ldots, X_m)'$ is (nondegenerate) *multivariate normal* if there exists an invertible $(m \times m)$ matrix $\mathbf{A}$ and a sequence of i.i.d. standard normal variates $Z_1, \ldots, Z_m$, such that $\mathbf{X} = \mathbf{A}'\mathbf{Z}$, where $\mathbf{Z} = (Z_1, \ldots, Z_m)'$, as a column vector. To compute its law, simply note that for all m-vectors $\boldsymbol{\xi}$ (written as a column vector),

$$\mathbb{E}\{e^{i\boldsymbol{\xi}'\mathbf{X}}\} = \mathbb{E}\{e^{i\boldsymbol{\xi}'\mathbf{A}'\mathbf{Z}}\} = \prod_{\ell=1}^{m} \mathbb{E}\{e^{i(\boldsymbol{\xi}'\mathbf{A}')_\ell Z_\ell}\} = e^{-\frac{1}{2}\sum_{\ell=1}^{m}(\boldsymbol{\xi}'\mathbf{A}')_\ell^2} = e^{-\frac{1}{2}\boldsymbol{\xi}'\mathbf{A}'\mathbf{A}\boldsymbol{\xi}}.$$

In fact, this makes perfect sense as long as $\mathbf{A}'\mathbf{A}$ is invertible, regardless of whether or not $\mathbf{A}$ is even a square matrix. Moreover, the density function of $\mathbf{X}$ is then given by

$$f(\mathbf{x}) = (2\pi)^{-\frac{d}{2}}\big(\det \mathbf{A}'\mathbf{A}\big)^{-\frac{1}{2}} \exp\big\{ -\tfrac{1}{2}\mathbf{x}'(\mathbf{A}'\mathbf{A})^{-1}\mathbf{x}\big\}, \qquad \forall \mathbf{x} \in \mathbb{R}^m.$$

From this formula, we immediately obtain that two Gaussian variables that are jointly Gaussian are independent *if and only if* their correlation is 0. Moreover, pairwise independence of jointly Gaussian random variables is equivalent to their total independence.

If T is an arbitrary index set, a *Gaussian process* $g = \{g_t;\ t \in T\}$ is a stochastic process such that for all finite $t_1, \ldots, t_m \in T$, the law of $(g_{t_1}, \ldots, g_{t_m})$ is multivariate Gaussian. (Remember that in these lectures, all Gaussian laws are centered, i.e., have mean 0.) Note that if such a process exists, its *covariance function* $(s,t) \mapsto \Sigma(s,t)$ completely determines its law, where

$$\Sigma(s,t) = \mathbb{E}\{g_s g_t\}, \qquad \forall s,t \in T.$$

It is easy to see that if g exists,

- Σ is a symmetric function.
  Proof. $\Sigma(s,t) = \mathbb{E}\{g_s g_t\} = \mathbb{E}\{g_t g_s\} = \Sigma(t,s)$. □

- Σ is a positive definite function.
  Proof. For all m and for all m-vectors η, and for all $s_1, \ldots, s_m \in T$,

$$\begin{aligned}\sum_{i=1}^m \sum_{j=1}^m \eta_i \eta_j \Sigma(s_i, s_j) &= \mathbb{E}\Big\{ \sum_{i=1}^m \sum_{j=1}^m \eta_i \eta_j g_{s_i} g_{s_j} \Big\} \\ &= \mathbb{E}\Big\{ \Big(\sum_{i=1}^m \eta_i g_{s_i} \Big)^2 \Big\},\end{aligned}$$

  which is positive.

Conversely, by the Daniell–Kolmogorov consistency theorem, any symmetric positive definite $\Sigma : T \times T \to \mathbb{R}$ corresponds to a Gaussian process, g, defined on the probability space $\mathbb{R}^T$ endowed with the product topology and the induced Borel field. This is not a good probability space, but is the best one can do in general. In any event, we now know that at least g exists! What it all amounts to is that once we have a symmetric, positive definite function Σ, it corresponds to a Gaussian process.

To define white noise, we only need to provide a formula for the covariance function Σ, and need to identify T. Let $T = \mathcal{B}(\mathbb{R}^N)$ denote the collection of all Borel measurable subsets of $\mathbb{R}^N$ of finite Lebesgue's measure. Moreover, we identify two elements, A_1 and A_2, of T if the set difference, $A_1 \triangle A_2$, has zero Lebesgue's measure.

Define

$$\Sigma(A, B) = |A \cap B|, \qquad \forall A, B \in \mathcal{B}(\mathbb{R}^N), \tag{1.1}$$

where $| \cdots |$ denotes the N-dimensional Lebesgue's measure. Clearly, Σ is symmetric. We seek to show that it is positive definite. But, for all $A_1, \ldots, A_m \in \mathcal{B}(\mathbb{R}^N)$, and all $\eta_1, \ldots, \eta_N \in \mathbb{R}$,

$$\begin{aligned}\sum_{i=1}^m \sum_{j=1}^m \Sigma(A_i, A_j) \eta_i \eta_j &= \sum_{i=1}^m \sum_{j=1}^m |A_i \cap A_j| \eta_i \eta_j \\ &= \sum_{i=1}^m \sum_{j=1}^m \int_{\mathbb{R}^N} \mathbf{1}_{A_i}(x) \mathbf{1}_{A_j}(x)\, dx\ \eta_i \eta_j \\ &= \int_{\mathbb{R}^N} \Big(\eta_i \sum_{i=1}^m \mathbf{1}_{A_i}(x) \Big)^2 dx,\end{aligned}$$

thus proving positive definiteness. This shows that

Theorem 1.1 *There exists a Gaussian process $\mathbb{W} = \{\mathbb{W}(A);\ A \in \mathcal{B}(\mathbb{R}^N)\}$ whose covariance function is described by Eq. (1.1).*

The process $\mathbb{W}$ is the famous *white noise* on $\mathbb{R}^N$. We begin studying its elementary properties.

Lemma 1.2 *If $A_1, A_2 \in \mathcal{B}(\mathbb{R}^N)$ are disjoint, $\mathbb{W}(A_1)$ and $\mathbb{W}(A_2)$ are independent. Moreover, $\mathbb{W}(A_1)$ is a mean zero normal variate with variance $|A_1|$.*

Proof For the first part, note that

$$\mathbb{E}\{\mathbb{W}(A_1)\mathbb{W}(A_2)\} = |A_1 \cap A_2| = 0.$$

Since uncorrelated Gaussian variables are independent, the first assertion follows. The second assertion follows from the definition. □

Next, consider nonrandom disjoint sets $A_1, A_2 \in \mathcal{B}(\mathbb{R}^N)$, and make a direct calculation to see that

$$\begin{aligned}\mathbb{E}\Big\{ \big[\mathbb{W}(A_1 \cup A_2) - \mathbb{W}(A_1) - \mathbb{W}(A_2)\big]^2 \Big\} &= \mathbb{E}\{[\mathbb{W}(A_1 \cup A_2)]^2\} + \mathbb{E}\{ [\mathbb{W}(A_1)]^2\} + \mathbb{E}\{[\mathbb{W}(A_2)]^2\} \\ &\quad + 2\mathbb{E}\{\mathbb{W}(A_1)\mathbb{W}(A_2)\} - 2\mathbb{E}\{\mathbb{W}(A_1 \cup A_2)\mathbb{W}(A_1)\} \\ &\quad - 2\mathbb{E}\{\mathbb{W}(A_1 \cup A_2)\mathbb{W}(A_2)\} \\ &= |A_1 \cup A_2| + |A_1| + |A_2| + 0 - 2|A_1| - 2|A_2|,\end{aligned}$$

thanks to Lemma 1.2 and the fact that A_1 and A_2 are disjoint. Using the latter property once more, we see that

$$A_1 \cap A_2 = \varnothing \implies \mathbb{E}\Big\{ \big[\mathbb{W}(A_1 \cup A_2) - \mathbb{W}(A_1) - \mathbb{W}(A_2)\big]^2 \Big\} = 0.$$

A similar calculation shows that for general $A_1, A_2 \in \mathcal{B}(\mathbb{R}^N)$,

$$\mathbb{W}(A_1 \cup A_2) = \mathbb{W}(A_1) + \mathbb{W}(A_2) - \mathbb{W}(A_1 \cap A_2), \qquad \text{a.s.}$$

One can extend this immediately to a finite number of A_i's by induction. However, we should recognize that there is a null set outside which the above fails, and this null set depends on the choice of the A_i's. In fact, it is *not* true that $\mathbb{W}$ is a random measure for almost every realization. However,

Lemma 1.3 *White noise is a vector-valued random measure on $\mathbb{R}^N$ in the sense of $L^2(\mathbb{P})$.*

Indeed, for this, you only need to check that when $A_1 \supseteq A_2 \supseteq \cdots$ are all in $\mathcal{B}(\mathbb{R}^N)$ and all have finite Lebesgue's measure, and if $\cap_n A_n = \varnothing$,

$$\lim_{n\to\infty} \mathbb{E}\big\{ \big[\mathbb{W}(A_n)\big]^2 \big\} = 0.$$

But this is easy.

2 Brownian Motion

Recall that $B = \{B(t);\, t \geq 0\}$ is *Brownian motion* if it is a Gaussian process on $\mathbb{R}$ with the covariance function

$$\Sigma(s,t) = \mathbb{E}\{B(s)B(t)\} = s \wedge t, \qquad \forall s,t \in \mathbb{R}_+.$$

To obtain this from white noise, let $\mathbb{W}$ denote white noise on $\mathbb{R}$ and *define*

$$X(t) = \mathbb{W}([0,t]), \qquad \forall t \geq 0.$$

Then, $X = \{X(s);\, s \geq 0\}$ is a Gaussian process with the same covariance function as Σ above. Thus, it is Brownian motion. In other words, to obtain properties of Brownian motion, we can assume that it is of form $\mathbb{W}([0,t])$. Since $\mathbb{W}$ is a kind of measure, this means that Brownian motion is the distribution function of white noise on $\mathbb{R}$, viewed as an $L^2(\mathbb{P})$-measure.

3 Čentsov's Representation

Recall that $B = \{B(\mathbf{s});\, s_1, s_2 \geq 0\}$ is *Brownian sheet* if it is a Gaussian process on $\mathbb{R}$ with the covariance function

$$\Sigma(\mathbf{s},\mathbf{t}) = \mathbb{E}\{B(\mathbf{s})B(\mathbf{t})\} = (s_1 \wedge t_1) \times (s_2 \wedge t_2), \qquad \forall \mathbf{s},\mathbf{t} \in \mathbb{R}^2_+.$$

Check that Brownian sheet can be realized as the distribution function of white noise on $\mathbb{R}^2$. That is, if $\mathbb{W}$ denotes white noise on $\mathbb{R}^2$, $\mathbf{t} \mapsto \mathbb{W}([0,t_1] \times [0,t_2])$ is Brownian sheet. This representation is due to Čentsov, and while it is simple, it has profound consequences; we will tap into some of them in the next lecture.

Lecture 2

Filtrations, Commutation, Dynamics

Now, we use Čentsov's representation to study how the process $\mathbf{t} \mapsto B(\mathbf{t})$ evolves near a given 'time point' $\mathbf{t} = (t_1, t_2)$. That is, given a fixed $\mathbf{t}$ with $t_1, t_2 > 0$, we wish to study the evolution of the 2-parameter process $\mathbf{s} \mapsto B(\mathbf{t} + \mathbf{s})$. You can think of the proceeding as 2-parameter Markov property.

Throughout, we realize $B(\mathbf{t})$ in its white noise formulation:

$$B(t_1, t_2) = \mathbb{W}([0, t_1] \times [0, t_2]), \qquad \forall \mathbf{t} \in \mathbb{R}^2_+.$$

Throughout, we will need the relation $\preccurlyeq$, defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by

$$\mathbf{s} \preccurlyeq \mathbf{t} \iff s_1 \leq t_1 \text{ and } s_2 \leq t_2, \qquad\qquad \forall \mathbf{s}, \mathbf{t} \in \mathbb{R}^2.$$

Note that $(\mathbb{R}^2, \preccurlyeq)$ is a partially order set. With this in mind, we can define a sequence of sigma-algebras $\mathcal{F} = \{\mathcal{F}(\mathbf{t});\ \mathbf{t} \in \mathbb{R}^2_+\}$ as follows: for all $\mathbf{t} \in \mathbb{R}^2_+$, $\mathcal{F}(\mathbf{t})$ denotes the sigma-algebra generated by the collection $\{B(\mathbf{s});\ (0,0) \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{t}\}$. This is a *filtration* with respect to $\preccurlyeq$ in the sense that

$$\mathbf{s} \preccurlyeq \mathbf{t} \implies \mathcal{F}(\mathbf{s}) \subseteq \mathcal{F}(\mathbf{t}).$$

Any sequence of sigma-algebras that is increasing with respect to $\preccurlyeq$ is called a *filtration*.

1 Commutation of the Brownian Filtration

We can always define the minimum operation $\curlywedge$ on $\mathbb{R}^2 \times \mathbb{R}^2$ by

$$(\mathbf{s} \curlywedge \mathbf{t})_i = \min(s_i, t_i), \qquad \forall \mathbf{s}, \mathbf{t} \in \mathbb{R}^2_+,\ i = 1, 2.$$

Given this definition, we say that any 2-parameter filtration $\mathcal{G}$ is *commuting* if for all $\mathbf{s}, \mathbf{t} \in \mathbb{R}^2_+$, $\mathcal{G}(\mathbf{s})$ and $\mathcal{G}(\mathbf{t})$ are conditionally independent given $\mathcal{G}(\mathbf{s} \curlywedge \mathbf{t})$. This means that for all bounded $\mathcal{G}(\mathbf{t})$-measurable random variables $\xi_{\mathbf{t}}$, all bounded $\mathcal{G}(\mathbf{s})$-measurable variates $\xi_{\mathbf{s}}$, we almost surely have

$$\mathbb{E}\{\xi_{\mathbf{t}} \times \xi_{\mathbf{s}} \mid \mathcal{G}(\mathbf{s} \curlywedge \mathbf{t})\} = \mathbb{E}\{\xi_{\mathbf{t}} \mid \mathcal{G}(\mathbf{s} \curlywedge \mathbf{t})\} \times \mathbb{E}\{\xi_{\mathbf{s}} \mid \mathcal{G}(\mathbf{s} \curlywedge \mathbf{t})\}.$$

Theorem 1.1 (R. Cairoli and J. B. Walsh) *The Brownian sheet filtration, $\mathcal{F}$, is commuting.*

Sketch of Proof Note that for any $\mathbf{t} \in \mathbb{R}^2_+$, $\mathcal{F}(\mathbf{t})$ is the sigma-field generated by $\{\mathbb{W}(A);\ A \subseteq [0,t_1] \times [0,t_2]\}$. Intuitively, this is because of the inclusion-exclusion formula of J. Poincaré. I will describe this in the simpler discrete setting. A formal verification is more difficult and requires more work, but few new ideas are needed; cf. MPP (Chapter 7, §4) for details.

In the discrete setting, we wish to construct white noise on $\mathbb{Z}^2$ (instead of $\mathbb{R}^2$). That is, we want

- for all $A \subseteq \mathbb{Z}^2$, $\mathbb{W}(A)$ is Gaussian with variance $\#A$;
- if $A_1, A_2 \subseteq \mathbb{Z}^2$ are disjoint, $\mathbb{W}(A_1)$ and $\mathbb{W}(A_2)$ are independent.

It is easy to construct such a white noise: simply let $\{\eta_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^2}$ be i.i.d. standard Gaussians, and *define*

$$\mathbb{W}(A) = \sum_{\mathbf{i}\in A} \eta_{\mathbf{i}}, \qquad \forall A \subset \mathbb{Z}^2.$$

(Check!) Discrete Brownian sheet is then $\mathbf{t} \mapsto \sum_{\mathbf{i} \preccurlyeq \mathbf{t}} \eta_{\mathbf{i}}$ for all $\mathbf{t} \in \mathbb{Z}^2$, and it follows (really from Poincaré's inclusion–exclusion formula) that if $\mathcal{F}(\mathbf{t})$ is the sigma-field generated by $\{B(\mathbf{s});\ \mathbf{s} \preccurlyeq \mathbf{t},\ \mathbf{s} \in \mathbb{Z}^2\}$, then $\mathcal{F}(\mathbf{t})$ is also the sigma-field generated by $\big\{\mathbb{W}(A);\ A \subseteq \big([0,t_1] \times [0,t_2]\big) \cap \mathbb{Z}^2\big\}$.

In continuous-time, one needs to be more careful, but this is the basic idea, nonetheless. $\square$

The above says alot about the evolution of the process $\mathbf{t} \mapsto B(\mathbf{t})$, and we will come back to it later. However, there are other evolutionary properties, as well. Here is one example.

Lemma 1.2 ($\preccurlyeq$–Markov property) *Fix $\mathbf{s} \in \mathbb{R}^2_+$ with $s_1, s_2 > 0$. Then, $\mathbf{t} \mapsto B(\mathbf{t}+\mathbf{s}) - B(\mathbf{s})$ is independent of $\mathcal{F}(\mathbf{s})$. In particular, conditional on $B(\mathbf{s})$, $\mathbf{t} \mapsto B(\mathbf{s}+\mathbf{t})$ is independent of $\mathcal{F}(\mathbf{s})$.*

In fact, we will soon see what the conditional distribution of the above process is.

Proof Clearly, whenever $\mathbf{u} \preccurlyeq \mathbf{s}$ are both in $\mathbb{R}^2_+$,

$$\begin{aligned}\mathbb{E}\big\{[B(\mathbf{s}+\mathbf{t}) - B(\mathbf{s})] \times B(\mathbf{u})\big\} &= \min(s_1+t_1, u_1) \times \min(s_2+t_2, u_2) - \min(s_1, u_1) \times \min(s_2, u_2) \\ &= u_1u_2 - u_1u_2 \\ &= 0.\end{aligned}$$

But for Gaussians, uncorrelatedness = independence. This shows that $\mathbf{t} \mapsto B(\mathbf{t}+\mathbf{s}) - B(\mathbf{s})$ is independent of $\mathcal{F}(\mathbf{s})$. The second assertion follows from the first. $\square$

But, $B(\mathbf{s}+\mathbf{t}) - B(\mathbf{s})$ is Gaussian with variance

$$\sigma^2(\mathbf{s},\mathbf{t}) = \mathbb{E}\{\left|\mathbb{W}([0,\mathbf{t}+\mathbf{s}] \setminus [0,\mathbf{s}])\right|^2\} = (t_1+s_1)(t_2+s_2) - s_1 s_2,$$

where $[0,\mathbf{u}] = [0,u_1] \times [0,u_2]$ for all $\mathbf{u} \in \mathbb{R}^2$. Thus, we can find the "law" of the "future" given $B(\mathbf{s})$:

$$\mathbb{E}\{f(B(\mathbf{s}+\mathbf{t})) \,|\, \mathcal{F}(\mathbf{s})\} = \frac{1}{\sqrt{2\pi\sigma^2(\mathbf{s},\mathbf{t})}} \int_{-\infty}^{+\infty} f(z + B(\mathbf{s})) \exp\left(-\frac{z^2}{2\sigma^2(\mathbf{s},\mathbf{t})}\right) dz. \tag{1.1}$$

2 Local Theory: Dynamics

We now wish to study the properties of the process near a given time point $\mathbf{s}$. For simplicity, let $\mathbf{1} = (1,1)$, $\mathbf{0} = (0,0)$, and consider the process $\mathbf{t} \mapsto B(\mathbf{t}+\mathbf{1}) - B(\mathbf{1})$. In white noise terms, this is

$$\begin{aligned} B(\mathbf{t}+\mathbf{1}) - B(\mathbf{1}) &= \mathbb{W}\big([\mathbf{0},\mathbf{t}+\mathbf{1}] \setminus [\mathbf{0},\mathbf{1}]\big) \\ &= \mathbb{W}\big([1,t_1+1] \times [0,1]\big) + \mathbb{W}\big([0,1] \times [1,t_2+1]\big) + \mathbb{W}\big([\mathbf{1},\mathbf{t}+\mathbf{1}]\big) \\ &:= \beta_1(t_1) + \beta_2(t_2) + B'(\mathbf{t}). \end{aligned}$$

The important thing to remember is that since whenever $|A_1 \cap A_2| = 0$, $\mathbb{W}(A_1)$ and $\mathbb{W}(A_2)$ are independent. This means that the processes β_1, β_2 and B' are all totally independent from one another, as well as $\mathcal{F}(\mathbf{1})$; the last statement comes from Lemma 1.2. On the other hand, for all $\mathbf{t}, \mathbf{s} \in \mathbb{R}^2_+$,

$$\begin{aligned} \mathbb{E}\{\beta_1(t_1) \cdot \beta_1(s_1)\} &= \mathbb{E}\Big\{\mathbb{W}\big([1,t_1+1] \times [0,1]\big) \cdot \mathbb{W}\big([1,s_1+1] \times [0,1]\big)\Big\} \\ &= \Big|\big([1,t_1+1] \times [0,1]\big) \cap \big([1,s_1+1] \times [0,1]\big)\Big| \\ &= \min(t_1, s_1). \end{aligned}$$

Thus, β_1 is a Brownian motion. By symmetry, β_2 is also a Brownian motion. Finally,

$$\begin{aligned} \mathbb{E}\big\{B'(\mathbf{s}) \cdot \mathbb{W}'(\mathbf{t})\big\} &= \Big|[\mathbf{1},\mathbf{t}+\mathbf{1}] \cap [\mathbf{1},\mathbf{s}+\mathbf{1}]\Big| \\ &= \min(s_1,t_1) \times \min(s_1,t_2). \end{aligned}$$

That is, $\mathbb{W}'$ is a Brownian sheet. We have proven the following:

Theorem 2.1 (W. Kendall) *The process $\mathbf{t} \mapsto B(\mathbf{t}+\mathbf{1})$ has the decomposition*

$$B(\mathbf{t}+\mathbf{1}) = B(\mathbf{1}) + \beta_1(t_1) + \beta_2(t_2) + B'(\mathbf{t}),$$

where β_1 and β_2 are Brownian motions, B' is a Brownian sheet, and (β_1, β_2, B') are entirely independent from one another, as well as from $B(\mathbf{1})$.

The above has been expanded upon very nicely in a series of articles by R. C. Dalang and J. B. Walsh; cf.. the Bibliography.

This decomposition is quite useful in analysing the sample paths of the sheet. For instance, suppose we are interested in the behavior of $B(\mathbf{t}+\mathbf{1})$ when $\mathbf{t} \approx \mathbf{0}$. Note that the variance of $\beta_1(t_1)$ ($\beta_2(t_2)$, resp.) is t_1 (t_2, resp.), while that of $B'(\mathbf{t})$ is $t_1 t_2$. Since $\mathbf{t} \approx \mathbf{0}$, it stands to reason that $B'(\mathbf{t})$ should be a.s. dominated by $\beta_1(t_1) + \beta_2(t_2)$, as $\mathbf{t} \to \mathbf{0}$. This can be made rigorous in various settings, and the end result, usually, is that, at least locally, one might expect

$$B(\mathbf{t}+\mathbf{1}) \approx B(\mathbf{1}) + \beta_1(t_1) + \beta_2(t_2).$$

The 2-parameter process $\mathbf{t} \mapsto \beta_1(t_1) + \beta_2(t_2)$ is *much* simpler to analyse, and is called *additive Brownian motion*. Of course, this discussion is heuristic, but the ideas introduced here can be useful in studying the local structure of the sheet, amongst other things.

SOMETHING TO TRY: Find a decomposition near a general point $\mathbf{s}$ with $s_1, s_2 > 0$ analogously. A much harder, though still possible, exercise is to completely characterize the process $\mathbf{t} \mapsto B(\mathbf{t})$ given $B(\mathbf{s})$ for a fixed $\mathbf{s} \in \mathbb{R}^2$ with $s_1, s_1 > 0$.

(HINT: $B(\mathbf{s}+\mathbf{t}) - B(\mathbf{s})$ should look like $s_2\beta_1(t_1) + s_1\beta_2(t_2)$, plus a Brownian sheet. For the rest of the decomposition, it suffice to consider the process $(t_1, t_2) \mapsto B(s_1 - t_1, s_2 + t_2)$ where $t_1 \in (0, s_1)$ and $t_2 \geq 0$. For this case, try finding $\gamma = \gamma_{\mathbf{s},\mathbf{t}}$ such that $\mathbf{t} \mapsto B(s_1 - t_1, s_2 + t_2) + \gamma B(\mathbf{s})$ is independent of $B(\mathbf{s})$. There is a unique choice of such a γ. Show that with this choice of γ, $\mathbf{t} \mapsto B(s_1 - t_1, s_2 + t_2) + \gamma B(\mathbf{s})$ is, in fact, independent of the sigma-algebra generated by $\{B(\mathbf{r});\ r_1 \geq s_1,\ 0 \leq r_2 \leq s_2\}$.)

Motivated by this heuristic discussion, we note that if $\mathbf{s} \in \mathbb{R}^2$ is fixed and if $s_1, s_2 > 0$, there exists a finite constant $c = c(\mathbf{s}) > 1$, such that for all $\mathbf{t} \in [\mathbf{s}, \mathbf{s}+\mathbf{1}]$,

$$\frac{1}{c}|\mathbf{t}| \leq \sigma^2(\mathbf{s},\mathbf{t}) \leq c|\mathbf{t}|. \tag{2.1}$$

This is motivated by the heuristics above, since at least for $\mathbf{t}$ small, $B(\mathbf{t}+\mathbf{s}) - B(\mathbf{s})$ is supposed to look like $s_2\beta_1(t_1) + s_1\beta_2(t_2)$ whose variance is exactly $s_2 t_1 + s_1 t_2$. The latter is between $\min(s_1, s_2)|t_1 + t_2|$ and $\max(s_1, s_2)|t_1 + t_2|$. Since all norms on $\mathbb{R}^2$ are equivalent, the displayed inequalities should follow.

SOMETHING TO TRY: Prove Eq. (2.1), either directly, or by appealing to a decomposition near $\mathbf{s}$ *à la* Theorem 2.1.

As a consequence of Eq. (2.1), used in conjunction with Eq. (1.1), we obtain the following analytical counterpart to our heuristic discussion about the sample paths of B near a point $\mathbf{s}$:

Lemma 2.2 *For each fixed* $\mathbf{s} \in \mathbb{R}^2_+$ *with* $s_1, s_2 > 0$, *there exists a constant* $C = C(\mathbf{s}) > 1$, *such that for all positive, measurable* $f : \mathbb{R} \to \mathbb{R}$, *and all* $\mathbf{t} \in \mathbb{R}^2_+$,

$$C^{-1} \int_{\mathbb{R}} f(z + B(\mathbf{s})) \frac{e^{-\frac{Cz^2}{|\mathbf{t}|}}}{\sqrt{|\mathbf{t}|}} \, dz \leq \mathbb{E}\big\{ f(B(\mathbf{s} + \mathbf{t})) \, \big| \, \mathcal{F}(\mathbf{s}) \big\} \leq C \int_{\mathbb{R}} f(z + B(\mathbf{s})) \frac{e^{-\frac{z^2}{C|\mathbf{t}|}}}{\sqrt{|\mathbf{t}|}} \, dz.$$

It is time to stop and see what we would have done had B been ordinary Brownian motion. In this case,

$$\begin{aligned} \mathbb{E}\{ f(B(t+s)) \, | \, \mathcal{F}(s) \} &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(z + B(s)) e^{-\frac{z^2}{2t}} \, dz \\ &= p_t \star f(B(s)) \\ &:= \mathfrak{S}_t f(B(s)), \end{aligned}$$

where $\star$ denotes convolution , and

$$p_t(x) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}, \qquad \forall t > 0, \ x \in \mathbb{R}.$$

By the Chapman-Kolmogorov equations, $\mathfrak{S}_{t+s} f(x) = \mathfrak{S}_t(\mathfrak{S}_s f)(x)$. In the language of operator theory, $\mathfrak{S}_{t+s} = \mathfrak{S}_t \mathfrak{S}_s$, which means that $\{\mathfrak{S}_t\}_{t \geq 0}$ is a convolution semigroup of linear operators. This is known as the *heat semigroup* and is intimately connected to parabolic PDE's based on the Laplacian Δ. Lemma 2.2 is a quantitative analogue of such operator-theoretic connections when there are two parameters involved. In disguise, it states that Brownian sheet is related to the two-parameter convolution semigroup $\mathbf{t} \mapsto \mathfrak{S}_{t_1} \mathfrak{S}_{t_2}$, but only in the sense of inequalities that hold "*locally*"; this is useful since $\mathfrak{S}_t$ is a positive operator in that "$f \geq 0$, a.e." $\Rightarrow$ "$\mathfrak{S}_t f \geq 0$, a.e." By "*local*", I mean that $\mathbf{s}$ is fixed, and we are looking locally around time $\mathbf{s}$. Finally, let me mention that such inequalities are the best that one can hope for, since it can be shown that exact connections to two-parameter semigroups do not hold in a useful and meaningful manner.

Another application of Eq. (2.1) is to the continuity of $\mathbf{t} \mapsto B(\mathbf{t})$. Note that

$$\begin{aligned} \mathbb{E}\{ |B(\mathbf{t} + \mathbf{s}) - B(\mathbf{s})|^2 \} &= \sigma^2(\mathbf{s}, \mathbf{t}) \\ &\leq c|\mathbf{t}|. \end{aligned}$$

But the following property of Gaussian random variables is easy to verify by direct calaculation: if g is a Gaussian variate, for any $p > 2$, there exists $\kappa(p)$ such that $\|g\|_p = \kappa(p) \|g\|_2^{p/2}$. Consequently,

$$\mathbb{E}\{ |B(\mathbf{t} + \mathbf{s}) - B(\mathbf{s})|^p \} \leq c\kappa(p) |\mathbf{t}|^{\frac{p}{2}}, \qquad \forall \mathbf{t} \in \mathbb{R}^2. \tag{2.2}$$

Now, we recall the following N-parameter formulation of Kolmogorov's continuity lemma that is proved as in the more familiar 1-parameter case.

Lemma 2.3 (A. N. Kolmogorov) *Let $x = \{x_{\mathbf{t}};\ \mathbf{t} \in \mathbb{R}^N\}$ be a random process indexed by $\mathbb{R}^N$, and suppose that for all compact $K \subset \mathbb{R}^N$, there exists $c, \varrho > 0$ and $\eta > N$ such that for all $\mathbf{s}, \mathbf{t} \in K$,*

$$\mathbb{E}\{|x_{\mathbf{t}+\mathbf{s}} - x_{\mathbf{s}}|^{\varrho}\} \leq c|\mathbf{t}|^{\eta}.$$

Then, x has a modification that is continuous.

The values of ϱ and c are immaterial to the content of this result. However, η must be *strictly* greater than N (the number of parameters) for this result to be applicable. Better results are possible via the notions of metric entropy, and majorizing measures, but the above is good enough for our purposes. Combined with our calculation for the Brownian sheet, Eq. (2.2), we obtain

Lemma 2.4 *Thre exists a modification of B that is continuous. In particular, one can construct the law $\mathbb{P} \circ B^{-1}$ on the space $C\big([0,\infty)^N\big)$ of continuous functions, endowed with the compact-open topology.*

The second line merely states that we do not have to construct the law of B on the ill-behaved space $\mathbb{R}^{\mathbb{R}_+^N}$ with product topology. But, rather, on $C\big([0,\infty)^N\big)$, which is quite a nice measure space. As a notational aside, recall that $\mathbb{P} \circ B^{-1}\{\bullet\}$ is the measure $\mathbb{P}\{B \in \bullet\}$.

Lecture 3

Cairoli's Theory of Martingales

Recall that $\mathbf{s} \preccurlyeq \mathbf{t}$ means that $s_1 \leq t_1$ and $s_2 \leq t_2$, and that $\mathbf{s} \curlywedge \mathbf{t} = (s_1 \wedge t_1, s_2 \wedge t_2) \in \mathbb{R}^2$. Recall also that $\mathcal{F} = \{\mathcal{F}(\mathbf{t});\ \mathbf{t} \in \mathbb{R}^2_+\}$ is a two-parameter filtration if (i) for each $\mathbf{t} \in \mathbb{R}^2_+$, $\mathcal{F}(\mathbf{t})$ is a sigma-algebra; and (ii) whenever $\mathbf{s} \preccurlyeq \mathbf{t}$ are both in $\mathbb{R}^2_+$, $\mathcal{F}(\mathbf{s}) \subseteq \mathcal{F}(\mathbf{t})$.

A 2-parameter process $\{M(\mathbf{t});\ \mathbf{t} \in \mathbb{R}^2_+\}$ is a *martingale* with respect to the filtration $\mathcal{F}$ if

(i) for each $\mathbf{t} \in \mathbb{R}^2_+$, $M(\mathbf{t})$ is $\mathcal{F}(\mathbf{t})$-measurable;

(ii) for each $\mathbf{t} \in \mathbb{R}^2_+$, $M(\mathbf{t}) \in L^1(\mathbb{P})$; and

(iii) whenever $\mathbf{s} \preccurlyeq \mathbf{t}$ are both in $\mathbb{R}^2_+$, $\mathbb{E}\{M(\mathbf{t}) \,|\, \mathcal{F}(\mathbf{s})\} = M(\mathbf{s})$, a.s.

In general, there is no useful theory of 2-parameter martingales, as there exist bounded 2-parameter martingales that do not converge; cf. Dubins and Pitman in the Bibliography, or Chapter 1 of MPP. However, Cairoli, and subsequently, Cairoli and Walsh have shown us that, under commutation, things work out rather nicely for 2-parameter (and, in general, multiparameter) martingales. In light of Theorem 1.1, we can apply such a martingale theory to the filtration of the Brownian sheet, which is our long-term goal. In this lecture, we mostly concentrate on aspects of the Cairoli-Walsh theory that we will need. In order to avoid the technical issues that come with continuous-time processes, we only consider martingales in discrete time. This will be ample for our needs. As such, throughout this lecture, our parameter set is some countable subset of $\mathbb{R}^2$ that inherits the partial order $\preccurlyeq$ as well. Without loss of much generality, we assume this to be $\mathbb{N}^2$, where $\mathbb{N} = \{1, 2 \ldots\}$ are the numerals.

1 Commutation and Conditional Independence

Recall that $\mathcal{F}$ is commuting if for all $\mathbf{s}, \mathbf{t} \in \mathbb{N}^2$, $\mathcal{F}(\mathbf{s})$ and $\mathcal{F}(\mathbf{t})$ are conditionally independent, given $\mathcal{F}(\mathbf{s} \curlywedge \mathbf{t})$.

Theorem 1.1 *If $\mathcal{F}$ is commuting, and V is a bounded random variable, for any $\mathbf{s} \in \mathbb{N}^2$,*

$$\mathbb{E}\{V \mid \mathcal{F}(\mathbf{s})\} = \mathbb{E}\Big\{\mathbb{E}[V \mid \mathcal{F}^1(s_1)] \,\Big|\, \mathcal{F}^2(s_2)\Big\}, \qquad \textit{a.s.},$$

where $\mathcal{F}^1(i) = \vee_{j \geq 1}\mathcal{F}(i,j)$ and $\mathcal{F}^2(j) = \vee_{i \geq 1}\mathcal{F}(i,j)$, $\forall i, j \geq 1$.

Henceforth, we call $\mathcal{F}^1 = \{\mathcal{F}^1(i);\ i \geq 1\}$ and $\mathcal{F}^2 = \{\mathcal{F}^2(j);\ j \geq 1\}$ the *marginal filtrations* of the 2-parameter filtration $\mathcal{F}$. Note that the marginal filtrations of a 2-parameter filtration are two 1-parameter filtrations in the usual sense.

Before proving Theorem 1.1, we establish a technical lemma.

Lemma 1.2 *A 2-parameter filtration $\mathcal{F}$ is commuting if and only if for all $\mathbf{s}, \mathbf{t} \in \mathbb{N}^2$, and for all bounded $\mathcal{F}(\mathbf{s})$-measurable variates $Y_{\mathbf{s}}$,*

$$\mathbb{E}\{Y_{\mathbf{s}} \mid \mathcal{F}(\mathbf{t})\} = \mathbb{E}\{Y_{\mathbf{s}} \mid \mathcal{F}(\mathbf{t} \curlywedge \mathbf{s})\}, \qquad \textit{a.s.}$$

Proof First, we suppose $\mathcal{F}$ is commuting. That is,

$$\mathbb{E}\{Y_{\mathbf{s}} \times Y_{\mathbf{t}} \mid \mathcal{F}(\mathbf{s} \curlywedge \mathbf{t})\} = \mathbb{E}\{Y_{\mathbf{s}} \mid \mathcal{F}(\mathbf{s} \curlywedge \mathbf{t})\} \times \mathbb{E}\{Y_{\mathbf{t}} \mid \mathcal{F}(\mathbf{s} \curlywedge \mathbf{t})\}, \qquad \text{a.s.}$$

Take expectations to see that

$$\begin{aligned}\mathbb{E}\{Y_{\mathbf{s}} \times Y_{\mathbf{t}}\} &= \mathbb{E}\Big[\mathbb{E}\{Y_{\mathbf{s}} \mid \mathcal{F}(\mathbf{s} \curlywedge \mathbf{t})\} \times \mathbb{E}\{Y_{\mathbf{t}} \mid \mathcal{F}(\mathbf{s} \curlywedge \mathbf{t})\}\Big] \\ &= \mathbb{E}\Big[Y_{\mathbf{t}} \times \mathbb{E}\{Y_{\mathbf{s}} \mid \mathcal{F}(\mathbf{s} \curlywedge \mathbf{t})\}\Big].\end{aligned}$$

Since this is true for all bounded $\mathcal{F}(\mathbf{t})$-measurable $Y_{\mathbf{t}}$, we have shown that commutation implies that for all bounded $\mathcal{F}(\mathbf{s})$-measurable variates $Y_{\mathbf{s}}$, $\mathbb{E}\{Y_{\mathbf{s}} \mid \mathcal{F}(\mathbf{t})\} = \mathbb{E}\{Y_{\mathbf{s}} \mid \mathcal{F}((\mathbf{s} \curlywedge \mathbf{t})\}$, almost surely. This is half the lemma. The converse half follows from the inclusion $\mathcal{F}(\mathbf{s} \curlywedge \mathbf{t}) \subseteq \mathcal{F}(\mathbf{s})$. $\square$

Proof of Theorem 1.1 In light of Lemma 1.2, Theorem 1.1 follows readily. Indeed, for all $i, j, n, m \geq 1$,

$$\begin{aligned}\mathbb{E}\Big[\overbrace{\mathbb{E}\{V \mid \mathcal{F}(i+n,j)\}}^{Y_{i+n,j}} \,\Big|\, \mathcal{F}(i,j+m)\Big] &= \mathbb{E}\big\{Y_{i+n,j} \,\big|\, \mathcal{F}(i,j)\big\} && \text{by Lemma 1.2,} \\ &= \mathbb{E}\{V \mid \mathcal{F}(i,j)\} && \text{since } \mathcal{F}(i,j) \subseteq \mathcal{F}(i+n,j).\end{aligned}$$

Now, let $m \uparrow \infty$ and use Doob's martingale convergence theorem to see that

$$\mathbb{E}\Big[\mathbb{E}\big\{V \,\big|\, \mathcal{F}(i+n,j)\big\} \,\Big|\, \mathcal{F}^1(i)\Big\} = \mathbb{E}\{V \mid \mathcal{F}(i,j)\}, \qquad \text{a.s.}$$

To finish, let $n \uparrow \infty$ and appeal to Doob's theorem once more. ☐

As a consequence of Theorem 1.1, we obtain the important maximal of R. Cairoli.

Theorem 1.3 (Cairoli's Maximal inequality) *Let $\mathcal{F}$ be a commuting filtration and consider a two-parameter martingale $M = \{M_{i,j};\, i,j \geq 1\}$.*

(i) *If $p > 1$, for all $n, m \geq 1$,*

$$\mathbb{E}\big\{\max_{(i,j)\preccurlyeq(n,m)} |M_{i,j}|^p\big\} \leq \big(\frac{p}{p-1}\big)^{2p}\, \mathbb{E}\{|M_{n,m}|^p\}.$$

(ii) *For $p = 1$, we have*

$$\mathbb{E}\big\{\max_{(i,j)\preccurlyeq(n,m)} |M_{i,j}|\big\} \leq \big(\frac{e}{e-1}\big)^2 \Big[1 + \mathbb{E}\{|M_{n,m}| \ln_+ |M_{n,m}|\}\Big].$$

I will only prove **(i)**, which is the part we need for these lectures.[†] When $p = 1$, things are only a little trickier; cf. MPP Chapter 1 for details.

Proof of (i) Note that for all $(i,j)\preccurlyeq(n,m)$, $M_{i,j} = \mathbb{E}\{M_{n,m} \mid \mathcal{F}(i,j)\}$, almost surely. Owing to Theorem 1.1,

$$M_{i,j} = \mathbb{E}\Big[\mathbb{E}\big\{M_{n,m} \,\big|\, \mathcal{F}^1(i)\big\} \,\Big|\, \mathcal{F}^2(j)\Big\}, \qquad \forall (i,j)\preccurlyeq(n,m), \text{ a.s.}$$

Consequently, by the conditional form of Jensen's inequality,

$$\max_{(i,j)\preccurlyeq(n,m)} |M_{i,j}|^p \leq \max_{j\leq m} \mathbb{E}\Big[\max_{i\leq n} \big|\overbrace{\mathbb{E}\big\{M_{n,m} \,\big|\, \mathcal{F}^1(i)\big\}}^{\Upsilon_i}\big|^p \,\Big|\, \mathcal{F}^2(j)\Big\}, \qquad \text{a.s.}$$

By Doob's maximal inequality for 1-parameter martingales, if $p > 1$,

$$\begin{aligned}\mathbb{E}\big\{\max_{(i,j)\preccurlyeq(n,m)} |M_{i,j}|^p\big\} &\leq \big(\frac{p}{p-1}\big)^p \max_{i\leq n} \mathbb{E}\{|\Upsilon_i|^p\} \\ &= \big(\frac{p}{p-1}\big)^p\, \mathbb{E}\Big[\max_{i\leq n} \big|\mathbb{E}\big\{M_{n,m} \,\big|\, \mathcal{F}^1(i)\big\}\big|^p\Big].\end{aligned}$$

[†] In fact, we will only need the $p = 2$ case.

Apply Doob's inequality again to finish. $\square$

SOMETHING TO TRY: Let $X_{i,j}$ be i.i.d. random variables with mean 0, and consider $S_{n,m} = \sum\sum_{(i,j)\preccurlyeq(n,m)} X_{i,j}$. Show that $S = \{S_{n,m};\ n, m \geq 1\}$ is a 2-parameter martingale with respect to a commuting filtration.

There is also a theory of 2-parameter reversed martingales that can be used to prove the following intriguing law of large numbers.

Theorem 1.4 (R. Smythe) *Let $X_{i,j}$ be i.i.d. random variables with mean μ, and consider the two-parameter random walk $S_{n,m} = \sum\sum_{(i,j)\preccurlyeq(n,m)} X_{i,j}$. Then,*

$$X_{1,1} \in L\log L(\mathbb{P}) \implies \mathbb{P}\Big\{\lim_{n,m\to\infty} \frac{S_{n,m}}{nm} = \mu\Big\} = 1, \text{ whereas}$$

$$X_{1,1} \notin L\log L(\mathbb{P}) \implies \mathbb{P}\Big\{\limsup_{n,m\to\infty} \frac{|S_{n,m}|}{nm} = +\infty\Big\} = 1.$$

SOMETHING TO TRY: Check, using the Borel–Cantelli lemma, that

$$X_{1,1} \notin L\log L(\mathbb{P}) \iff \mathbb{P}\Big\{\limsup_{n,m\to\infty} \frac{|X_{n,m}|}{nm} = +\infty\Big\} = 1.$$

Show that this implies the second half of Smythe's theorem.

I will append Chapter 2 of MPP to illustrate two examples of martingale theory in classical analysis: one to differentiation theory (the differentiation theorem of Lebesgue, as well as that of Jessen, Marcinkiewicz and Zygmund), and another to the the Haar function expansion of the elements of $L^1([0,1]^N)$.

Lecture 4

Capacity, Energy and Dimension

We now come to the second part of these lectures which has to do with "exceptional sets". The most obvious class of exceptional sets are those of measure 0, where the measure is some nice one. As an example, consider a compact set $E \subset \mathbb{R}^d$. One way to construct its Lebesgue measure is as follows: cover E by small boxes, compute the volume of the cover, and then optimize over all the covers. That is,

$$|E| = \lim_{\varepsilon \to 0^+} \inf \Big\{ \sum_i [\mathrm{diam}(E_i)]^d : \ E_1, E_2, \ldots \text{ closed boxes of diameter} \le \varepsilon \text{ with } \cup_i E_i \supseteq E \Big\}.$$

Here, we are computing the diameter of the box as twice its ℓ^1-radius; i.e., it is the length of any side. This is equivalent to the usual definition of Lebesgue's measure, although it is long out of fashion in standard analysis courses.

1 Hausdorff Dimension and Measures

The first class of exceptional sets that we can discuss are those of Lebesgue's measure 0, of course. But, this is too crude for differentiating amongst very thin sets. For example, consider the rationals $\mathbb{Q}$, as well as Cantor's tertiary set $\mathbf{C}$. While they are both measure 0 sets, $\mathbf{C}$ is uncountable, whereas $\mathbb{Q}$ is not. We would like a concrete way of saying that $\mathbf{C}$ is larger than $\mathbb{Q}$, and perhaps measure how much larger, as well. There are many ways of doing this, and we will choose a route that is useful for our probabilistic needs. First, note that for any $\alpha \ge 0$, we can define the analogue of $|E|$ as above. Namely, define for any compact set $E \subset \mathbb{R}^d$,

$$\mathcal{H}_\alpha(E) = \lim_{\varepsilon \to 0^+} \inf \Big\{ \sum_i [\mathrm{diam}(E_i)]^\alpha : \ E_1, E_2, \ldots \text{ closed boxes of diameter} \le \varepsilon \text{ with } \cup_i E_i \supseteq E \Big\}.$$

This makes sense even if $\alpha \le 0$.

The set function $\mathcal{H}_\alpha$ is called the α-dimensional *Hausdorff measure*. This terminology is motivated by the following, which is proved by using the method given to us by Carathéodory:

Theorem 1.1 *The set function $\mathcal{H}_\alpha$ is an outer measure on Borel subsets of $\mathbb{R}^d$. For all $\alpha > d$, $\mathcal{H}_\alpha(E) = 0$ identically. On the other hand, when $\alpha \leq d$ is an integer, $\mathcal{H}_\alpha(E)$ equals the α-dimensional Lebesgue's measure of Borel set E.*

Remark For the above to hold, we have used ℓ^∞ balls (i.e., boxes). If you use ℓ^2-balls in the definition instead, you will see that for integral α, $\mathcal{H}_\alpha$ equals ω_α times α-dimensional Lebesgue's measure, where ω_α is the volume of an α-dimensional ball of radius 1. $\square$

Hausdorff dimensions provide us with a more refined sense of how big a set is. Note that for any compact (or even Borel, say) set E, there is *always* a critical α such that for all $\beta < \alpha$, $\mathcal{H}_\beta(E) = 0$, while for all $\beta > \alpha$, $\mathcal{H}_\beta(E) = +\infty$. This is an easy calculation. But it leads to the following important measure-theoretic notion of dimension:

$$\dim(E) = \inf\{\alpha : \mathcal{H}_\alpha(E) = 0\} = \sup\{\alpha : \mathcal{H}_\alpha(E) = +\infty\}.$$

This is the *Hausdorff dimension* of E. If $E \subset \mathbb{R}^d$ is not compact, define $\dim(E)$ as $\sup_{n\geq 1} \dim(E \cap [-n, n]^d)$.

How does one compute the Hausdorff dimension of a set? You typically proceed by establishing an upper bound, as well as a lower bound. The first step is not hard: just find a "good" covering E_i of diameter less than ε, and compute $\sum_i [\text{diam}(E_i)]^\alpha$. Here is one way to get an upper bound systematically; other ways abound.

Suppose we are interested in computing the Hausdorff dimension of a given compact set $E \subset [0, 1]^d$. Fix a *real* number $n \geq 1$, and define $E_j = [\frac{j}{n}, \frac{j+1}{n}[$, for integers $0 \leq j \leq n$. Then, it is clear that the diameter of each E_j is no more than $\frac{2}{n}$, while $\cup_j E_j \supset E$. So,

$$\mathcal{H}_\alpha(E) \leq \big(\frac{2}{n}\big)^\alpha \mathcal{N}_n(E),$$

where $\mathcal{N}_n(E) = \sum_{0\leq j\leq n} \mathbf{1}\{I_{j,n} \cap E \neq \varnothing\}$ is the number of times the intervals $I_{j,n}$ contains portions of E. Therefore, if we can find α such that $\limsup_n n^{-\alpha}\mathcal{N}_n(E) < +\infty$, we have $\dim(E) \leq \alpha$.[†] Incidentally, the minimal α such that $\limsup_n n^{-\alpha}\mathcal{N}_n(E) < +\infty$ is the so-called *upper Minkowski (or box) dimension* of E. If we write the latter as $\dim_{_M}(E)$, we have shown that

$$\dim(E) \leq \dim_{_M}(E). \tag{1.1}$$

If we replace E_j by a d-dimensional box of the form $[\frac{i_1}{n}, \frac{i_1+1}{n}[\times \cdots \times [\frac{i_d}{n}, \frac{i_d+1}{n}[$ and repeat the procedure, we obtain the upper Minkowski dimension in d dimensions, and Eq. (1.1) remains to hold.

[†]We do not require n to be an integer here.

We now use this to obtain an upper bound for the tertiary Cantor set $\mathbf{C}$. First, let us recall the following iterative construction of $\mathbf{C}$: let $\mathbf{C}_0 = [0,1]$. Now, remove the middle third to obtain $\mathbf{C}_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Next, remove the middle thirds of each of the two subintervals to get $\mathbf{C}_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and so on. In this way, you have a decreasing sequence of compact subsets of $[0,1]$, and, as such, $\mathbf{C} = \cap_n \mathbf{C}_n$ is a nontrivial compact subset of $[0,1]$. At the nth level of construction, $\mathbf{C}_n$ is comprised of 2^n intervals of length 3^{-n}. Therefore, $|\mathbf{C}_n| = (\frac{2}{3})^n \to |\mathbf{C}| = 0$. On the other hand, we just argued that there are 2^n boxes, of diameter no greater than (in fact, equal to) 3^n that cover $\mathbf{C}$. Therefore, we have shown that $\mathcal{N}_{3^n}(\mathbf{C}) = 2^n$. In particular, for any $\alpha > \log_3(2)$, $\limsup_{m\to\infty}(3^{-m})^{-\alpha}\mathcal{N}_{3^m}(E) = \lim_{m\to\infty} 3^{-m\alpha}2^m = 0$. So that, after a little work, we get $\dim_{_M}(E) \leq \log_3(2)$. In fact, it is easy to see, by the same reasoning, that $\dim_M(E) = \log_3(2)$. In any event, we obtain the following:

$$\dim(\mathbf{C}) \leq \log_3(2) = \frac{\ln 2}{\ln 3}. \tag{1.2}$$

We will show that this is sharp in that the above inequality is an equality. But first, a question: *why not stick to Minkowski dimension?* It is certainly easier to compute than Hausdorff dimension, and at first sight, more natural. To answer this, try computing $\dim_{_M}(\mathbb{Q})$, or $\dim_{_M}$ of any other dense subset of $[0,1]^d$ for that matter! You will see that the answer is 1! On the other hand, it is not hard to show that $\dim(E) = 0$ if E is countable, for then we can write $E = \{r_i\}$ and note that $\{r_i\}$ is a cover of E with diameter less than ε. This seemingly technical difference is really a big one.

Now, to the lower bound for $\dim(\mathbf{C})$. Obtaining lower bound on Hausdorff dimension is, in principle, very hard, since you have to work uniformly over all covers. What makes things difficult is that there are *alot* of potential covers!

The ingenious idea behind obtaining lower bounds is due to O. Frostman who found it in his Ph.D. thesis in the 1935! Namely,

Theorem 1.2 (Frostman's lemma) *Suppose we knew that the compact set E carries a probability measure μ that is Hölder-smooth in the following sense: there exists $\alpha > 0$ and a constant C such that for all $r \in (0,1)$,*

$$\mu(\mathcal{B}(y,r)) \leq Cr^\alpha,$$

for μ-almost all y, where $\mathcal{B}(y,r)$ is the ℓ^∞-ball of radius r about $y \in \mathbb{R}^d$. Then, $\dim(E) \geq \alpha$.

There is a converse to this that we will only need once, and will not prove, as a result; for a proof, see Appendix C of MPP.

Theorem 1.3 (Frostman's Lemma (continued)) *Suppose* $\dim(E) \geq \alpha > 0$*. Then, for each* $\beta < \alpha$*, there exists* $\mu \in \mathcal{P}(E)$ *such that*

$$\sup_{x \in \mathbb{R}^d} \sup_{r \in (0,1)} \frac{\mu\{\mathcal{B}(x,r)\}}{r^\beta} < +\infty.$$

Proof I will prove this when instead of μ-almost all x, the lemma holds for all x. The necessary modifications to prove the general case are technical but not hard.

Fix $\varepsilon \in (0,1)$, and consider any cover $E_1, E_2, \ldots$ of diameter $\leq \varepsilon$. Note that

$$1 = \mu(E) \leq \sum_i \mu(E_i) \leq C \sum_i \left[\text{diam}(E_j)\right]^\alpha.$$

Optimize over all such covers, and let $\varepsilon \to 0$, to see that $1 \leq 2C\mathcal{H}_\alpha(E)$. The theorem follows, since this shows that for any $\beta < \alpha$, $\mathcal{H}_\beta(E) = +\infty$. (To prove in the general case, note that if $\mu(E_j)$ is not less than $C[\text{diam}(E_j)]^\alpha$, we can cover E_j by at most 2^d compact intervals $F_{j,1}, \ldots, F_{j,2^d}$ of diameter less than twice that of E_j, such that $\mu(F_{j,k}) \leq C[\text{diam}(F_{j,k})]^\alpha \leq 2^\alpha C[\text{diam}(E_j)]^\alpha$. Thus, $\mu(E_j) \leq 2^{\alpha+d} C[\text{diam}(E_j)]^\alpha$, which is good enough.) $\square$

We use this to complete our proof of the following.

Proposition 1.4 *If* $\mathbf{C}$ *denotes the tertiary Cantor set,* $\dim(\mathbf{C}) = \frac{\ln 2}{\ln 3}$.

Proof In light of what we have already done, we only need to verify the lower bound on dimension. We do this by finding a sufficiently smooth measure on $\mathbf{C}$. Our choice is more or less obvious and is found iteratively as follows: construct the smoothest possible probability measure μ_n on $\mathbf{C}_n$ and "take limits". Now, the smoothest and flattest probability measure on $\mathbf{C}_n$ is the uniform measure, μ_n. It is easy to see that for all $x \in [0,1]$,

$$\mu_n\big([x - 3^{-n}, x + 3^{-n}]\big) \leq 2^{-n} = (3^{-n})^{\ln 2/\ln 3}. \tag{1.3}$$

This is suggestive, but we need to work a little bit more. To do so, we next note that the μ_n's are nested: We write $\mathbf{C}_n = \cup_{i=1}^{2^n} I_{i,n}$ where $I_{i,n}$ is an interval of length 3^{-n}. The nested property of the μ_n's is the following, which can be checked by induction:

$$\forall n \geq m, \forall j = 1, \ldots, 2^m: \qquad \mu_n(I_{j,m}) = \mu_m(I_{j,m}) = 2^{-m}.$$

Standard weak convergence theory guarantees us of the existence of a probability measure μ_∞ on the compact set $\mathbf{C}$ such that for all $m \geq 1$ and all $j = 1, \ldots, 2^m$,

$$\mu_\infty(I_{j,m}) = \mu_m(I_{j,m}) = 2^{-m}.$$

Moreover, Eq. (1.3) extends to μ_∞. Namely, for all $x \in [0,1]$ and all $n \geq 0$,

$$\mu_\infty([x - 3^{-n}, x + 3^{-n}]) \leq (3^{-n})^{\ln 2/\ln 3}.$$

Now, if $r \in (0,1)$, we can find $n \geq 0$ such that $3^{-n-1} \leq r \leq 3^{-n}$. Therefore,

$$\sup_x \mu_\infty([x-r, x+r]) \leq \sup_x \mu_\infty([x - 3^{-n}, x + 3^{-n}]) \leq (3^{-n})^{\ln 2/\ln 3} \leq (3r)^{\ln 2/\ln 3}.$$

So, we have found a probability measure μ_∞ on $\mathbf{C}$, that satisfies the condition of Frostman's lemma with $C = 3^{\ln 2/\ln 3} = 2$ and $\alpha = \ln 3/\ln 3$. This completes our proof. □

2 Energy and Capacity

Suppose μ is a probability measure on some given compact set $E \subset \mathbb{R}^d$. We will write this as $\mu \in \mathcal{P}(E)$, and define for any measurable function $f : E \times E \to \mathbb{R}_+ \cup \{\infty\}$,

$$\mathcal{E}_f(\mu) = \iint f(x,y)\, \mu(dx)\, \mu(dx).$$

This is the *energy* of μ with respect to the gauge function f; it is always defined although it may be infinite. The following energy forms are of use to us:

$$\mathsf{Energy}_\alpha(\mu) = \iint |x-y|^{-\alpha}\, \mu(dx)\, \mu(dy),$$

where $|x| = \max_{1 \leq j \leq d} |x_j|$ for concreteness, although any other Euclidean norm will do just as well. This is the so-called α-dimensional *Bessel–Riesz energy* of μ. The question, in the flavor of the previous section, is *when does a set E carry a probability measure of finite energy?* To facilitate the discussion, we define the *capacity* of a set E by

$$\mathcal{C}_f(E) = \Big[\inf_{\mu \in \mathcal{P}(E)} \mathcal{E}_f(\mu) \Big]^{-1}, \qquad \text{and in particular,}$$
$$\mathsf{Cap}_\alpha(E) = \Big[\inf_{\mu \in \mathcal{P}(E)} \mathsf{Energy}_\alpha(\mu) \Big]^{-1}.$$

The above is Gauss' principle of minimum energy. Next, we argue that there is a minimum energy measure called the equilibrium measure. Moreover, its potential is essentially constant, and the constant is the energy.

Theorem 2.1 (Equilibrium Measure) *Suppose E is a compact set in $\mathbb{R}^d$ such that for some $\alpha > 0$, $\mathsf{Cap}_\alpha(E) > 0$. Then, there exists $\mu \in \mathcal{P}(E)$, such that*

$$\mathsf{Energy}_\alpha(\mu) = \big[\mathsf{Cap}_\alpha(E)\big]^{-1}.$$

Moreover, for μ-almost all x,

$$\int |x-y|^{-\alpha}\,\mu(dy) = \mathsf{Energy}_\alpha(\mu).$$

Proof By definition, there exists a sequence of probability measures μ_n, all supported on E, such that (i) they have finite energy; and (ii) for all $n \geq 1$, $(1+\frac{1}{n})[\mathsf{Cap}_\alpha(E)]^{-1} \geq \mathsf{Energy}_\alpha(\mu_n) \geq [\mathsf{Cap}_\alpha(E)]^{-1}$. Let μ be any subsequential limit of the μ_n's. Since $\mu \in \mathcal{P}(E)$ as well, $\mathsf{Energy}_\alpha(\mu) \geq [\mathsf{Cap}_\alpha(E)]^{-1}$. We aim to show the converse holds too. By going to a subsequence n' along which $\mu_{n'}$ converges weakly to μ, we see that for any $r_0 > 0$,

$$\iint_{|x-y|\geq r_0} |x-y|^{-\alpha}\,\mu(dx)\,\mu(dy) = \lim_{n'\to\infty}\iint_{|x-y|\geq r_0} |x-y|^{-\alpha}\,\mu_{n'}(dx)\,\mu_{n'}(dy) \leq [\mathsf{Cap}_\alpha(E)]^{-1}.$$

Let $r_0 \downarrow 0$ and use the dominated convergence theorem to deduce the first assertion. For the second assertion, i.e., that the minimum energy principle is actually achieved for some probability measure.

Now, consider

$$\Upsilon_\eta = \Big\{x \in E : \int |x-y|^{-\alpha}\,\mu(dy) < (1-\eta)\mathsf{Energy}_\alpha(\mu)\Big\}, \qquad \eta \in (0,1).$$

We wish to show that $\mu(\Upsilon_\eta) = 0$ for all $\eta \in (0,1)$. If this is not the case for some $\eta \in (0,1)$, then, consider the following

$$\zeta(\bullet) = \frac{\mu(\bullet \cap \Upsilon_\eta)}{\mu(\Upsilon_\eta)}.$$

Evidently, $\zeta \in \mathcal{P}(E)$, and has finite energy. Define

$$\lambda_\varepsilon = (1-\varepsilon)\mu + \varepsilon\zeta, \qquad \varepsilon \in (0,1).$$

Then, λ_ε is also a probability measure on E, and it, too, has finite energy. In fact, writing $\lambda_\varepsilon = \mu - \varepsilon(\mu-\zeta)$, a little calculation shows that

$$\mathsf{Energy}_\alpha(\lambda_\varepsilon) = \mathsf{Energy}_\alpha(\mu) + \varepsilon^2\mathsf{Energy}_\alpha(\mu-\zeta) - 2\varepsilon\iint |x-y|^{-\alpha}\,\mu(dx)\,\big[\mu(dy) - \zeta(dy)\big].$$

(The energy of $\mu - \zeta$ is defined as if $\mu - \zeta$ were a positive measure.)

Since μ minimizes energy, the above is greater than or equal to $\mathsf{Energy}_\alpha(\mu)$. Thus,

$$\varepsilon^2 \mathsf{Energy}_\alpha(\mu - \zeta) \geq 2\varepsilon \iint |x-y|^{-\alpha}\, \mu(dx) \left[\mu(dy) - \zeta(dy)\right].$$

Divide by ε and let $\varepsilon \to 0$ to see that

$$\mathsf{Energy}_\alpha(\mu) \leq \iint |x-y|^{-\alpha}\, \mu(dx)\, \zeta(dy).$$

But by the definition of Υ_η, the right hand side is no more than $(1-\eta)\mathsf{Energy}_\alpha(\mu)$, which contradicts the assumption that $\mu(\Upsilon_\eta) > 0$. In other words,

$$\int |x-y|^{-\alpha}\, \mu(dy) \geq \mathsf{Energy}_\alpha(\mu), \qquad \mu\text{-a.s.}$$

It suffices to show the converse inequality. But this is easy. Indeed, suppose

$$\mathfrak{G}\mu(x) = \int |x-y|^{-\alpha}\, \mu(dy) \geq (1+\eta)\mathsf{Energy}_\alpha(\mu),$$

on a set of positive μ-measure. The function $x \mapsto \mathfrak{G}\mu(x)$ is the *α-dimensional potential* of the measure μ. We could integrate $[d\mu]$ to get the desired contradiction, viz.,

$$\begin{aligned}
\mathsf{Energy}_\alpha(\mu) &= \int_{\Theta_\eta} \mathfrak{G}\mu(x)\, \mu(dx) + \int_{\Theta_\eta^\complement} \mathfrak{G}\mu(x)\, \mu(dx) \\
&\geq (1+\eta)\mathsf{Energy}_\alpha(\mu) \cdot \mu(\Theta_\eta) + \int_{\Theta_\eta^\complement} \mathfrak{G}\mu(x)\, \mu(dx),
\end{aligned}$$

where $\Theta_\eta = \{x: \ \mathfrak{G}\mu(x) \geq (1+\eta)\mathsf{Energy}_\alpha(\mu)\}$. Therefore, by Theorem 2.1 on equilibrium measure,

$$\begin{aligned}
\mathsf{Energy}_\alpha(\mu) &\geq \mathsf{Energy}_\alpha(\mu)\left[(1+\eta)\mu(\Theta_\eta) + \mu(\Theta_\eta^\complement)\right] \\
&= \mathsf{Energy}_\alpha(\mu)\left[1 + \eta\mu(\Theta_\eta^\complement)\right],
\end{aligned}$$

which is a contradiction, unless $\mu(\Theta_\eta) = 0$. This concludes our proof. $\square$

SOMETHING TO TRY: The α-dimensional Bessel–Riesz energy defines a Hilbertian pre-norm. Indeed, define $\mathcal{M}_\alpha(E)$ to be the collection of all measures of finite α-dimensional Bessel–Riesz energy on E. On this, define the inner product,

$$\langle \mu, \nu \rangle = \iint |x-y|^{-\alpha}\, \mu(dx)\, \nu(dy).$$

Check that this defines a positive-definite bilinear form on $\mathcal{M}_\alpha(E)$ if $\alpha \in (0, d)$. From this, conclude that for all $\mu, \nu \in \mathcal{M}_\alpha(E)$, $\langle \mu, \nu \rangle^2 \leq \mathsf{Energy}_\alpha(\mu) \cdot \mathsf{Energy}_\alpha(\nu)$. This fills a gap in the above proof.

The *capacitary dimension* of a compact set $E \subset \mathbb{R}^d$ is defined as

$$\dim_c(E) = \sup\left\{\alpha : \mathsf{Cap}_\alpha(E) > 0\right\} = \inf\left\{\alpha : \mathsf{Cap}_\alpha(E) = 0\right\}.$$

Theorem 2.2 (Frostman's Theorem) *Capacitary and Hausdorff dimensions are one and the same.*

Proof Here is one half of the proof: we will show that if there exists $\alpha > 0$ and a probability measure μ on E, such that $\mathsf{Energy}_\alpha(\mu) < +\infty \;\Rightarrow\; \dim(E) \geq \alpha$. This shows that $\dim_c(E) \leq \dim(E)$, which is half the theorem.

By Theorem 2.1, we can assume without loss of generality that μ is an equilibrium measure. In particular,

$$\mu(\mathcal{B}(x,r)) \leq r^\alpha \int |x-y|^{-\alpha}\,\mu(dy) = r^\alpha \mathsf{Energy}_\alpha(\mu),$$

μ-almost everywhere. Frostman's lemma (Theorem 1.2) shows that $\dim(E) \geq \alpha$, as needed.

For the other half, we envoke the second half of Frostman's theorem (Theorem 1.3) to produce for each $\beta < \dim(E)$ a probability measure $\mu \in \mathcal{P}(E)$, such that

$$\mu(\mathcal{B}(x,r)) \leq C r^\beta, \qquad \forall x \in \mathbb{R}^d,\ r \in (0,1).$$

But if D denotes the diameter of E,

$$\begin{aligned}
\mathsf{Energy}_\gamma(\mu) &= \sum_{j=0}^\infty \iint_{2^{-j-1}D \leq |x-y| \leq 2^{-j}D} |x-y|^{-\gamma}\,\mu(dx)\,\mu(dy) \\
&\leq \sum_{j=0}^\infty 2^{(j+1)\gamma} D^{-\gamma} \sup_{x \in \mathbb{R}^d} \mu(\mathcal{B}(x, 2^{-j}D) \\
&\leq C 2^\gamma D^{\beta-\gamma} \sum_{j=0}^\infty 2^{j\gamma} 2^{-j\beta},
\end{aligned}$$

which sums if $\gamma < \beta$. Thus, we have shown that for all $\gamma < \dim(E)$, $\mathsf{Cap}_\gamma(E) > 0$, i.e., $\dim_c(E) \geq \gamma$ for all $\gamma < \dim(E)$, which completes the proof. $\square$

3 The Brownian Curve

Next, we roll up our sleeves and compute the Hausdorff dimension of a few assorted and interesting random fractals that arise from Brownian considerations. Our goal is to illustrate the methods and ideas rather than the final word on this subject.

Throughout, $B = \{B_t;\, t \geq 0\}$ denotes Brownian motion in $\mathbb{R}^d$. Recall also that B is a strong Markov process, and that

$$B \text{ hits points iff } d = 1, \text{ i.e., } \exists t > 0: \ B_t = 0 \iff d = 1.$$

In particular, note that when $d = 1$, the Brownian curve has full Lebesgue measure, and also full dimension. On the other hand, when $d \geq 2$, the Brownian curve has zero Lebesgue measure (Lévy's theorem), despite the following result.

Theorem 3.1 *If B denotes d-dimensional Brownian motion, where $d \geq 2$, $\dim B(\mathbb{R}_+) = 2$, a.s.*

Proof We do this in two parts. First, we show that $\dim B(\mathbb{R}_+) \leq 2$ (*the upper bound*), and then we show that $\dim B(\mathbb{R}_+) \geq 2$ (*the lower bound*). In any event, recall that $d \geq 2$.

Proof of the upper bound Recall that for any interval $I \subset \mathbb{R}^d$,

$$\mathbb{P}\{B[1,2] \cap I \neq \varnothing\} \leq c\kappa(|I|), \text{ where } \kappa(\varepsilon) = \begin{cases} \varepsilon^{d-2}, & \text{if } d \geq 3 \\ \ln_+\left(\frac{1}{\varepsilon}\right), & \text{if } d = 2 \end{cases}. \tag{3.1}$$

We will obtain this sort of estimate, in the more interesting case of Brownian sheet, in the last lecture. In fact, one can show that the constant c depends only on M, as long as $I \subseteq [-M, M]^d$. Consider $I_1, \ldots, I_{n^d}$ cubes of side $\frac{1}{n}$, such that (i) $I_i^\circ \cap I_j^\circ = \varnothing$ if $i \neq j$; and (ii) $\cup_{j=1}^{n^d} I_j = [0,1]^d$. Based on these, define

$$E_j = \begin{cases} I_j, & \text{if } I_j \cap B[1,2] \neq \varnothing \\ \varnothing, & \text{otherwise} \end{cases}.$$

Note that $E_1, \ldots, E_{n^d}$ is a $(\frac{1}{n})$-cover of $B[1,2] \cap [0,1]^d$. Thus,

$$\mathcal{H}_\alpha\big(B[1,2] \cap [0,1]^d\big) \leq \liminf_{n\to\infty} \sum_{j=1}^{n^d} n^{-\alpha} \mathbf{1}_{\{I_j \cap B[1,2] \neq \varnothing\}}.$$

Consequently, as long as $\alpha > 2$,

$$\mathbb{E}\big\{\mathcal{H}_\alpha\big(B[1,2] \cap [0,1]^d\big)\big\} \leq c \liminf_{n\to\infty} \sum_{j=1}^{n^d} n^{-\alpha} \kappa\big(\tfrac{1}{n}\big) = c \liminf_{n\to\infty} n^{d-\alpha} \kappa\big(\tfrac{1}{n}\big) = 0.$$

In particular, $\dim(B[1,2] \cap [0,1]^d) \leq 2$, a.s. Similarly, $\dim(B[a,b] \cap [-n,n]^d) \leq 2$, a.s. for any $0 < a < b$ and $n > 0$. Let $n \uparrow \infty$, $a \downarrow 0$ and $b \uparrow \infty$, all along rational sequences to deduce that $\dim B(\mathbb{R}_+) \leq 2$, a.s. This uses the easily verified fact that whenever $A_1 \subseteq A_2 \subseteq \cdots$ are compact, and if $\mathcal{H}_\alpha(A_j) = 0$, then $\mathcal{H}_\alpha(\cup_j A_j) = 0$.

Proof of the lower bound For the converse, we will show that $\dim B[1,2] \geq 2$, and do this by appealing to Frostman's theorem (Theorem 2.2). To do so, we need to define a probability, or at least a finite, measure on the Brownian curve. The most natural measure that lives on the curve of $\{B_s; 1 \leq s \leq 2\}$ is the occupation measure:

$$\mathbb{O}(E) = \int_1^2 \mathbf{1}_{\{B_s \in E\}}\, ds.$$

With this in mind, note that for any $\alpha > 0$,

$$\mathsf{Energy}_\alpha(\mathbb{O}) = \iint |x-y|^{-\alpha}\, \mathbb{O}(dx)\, \mathbb{O}(dy) = \int_1^2 \int_1^2 |B_s - B_t|^{-\alpha}\, ds\, dt.$$

By Frostman's theorem, it suffices to show that $\mathbb{E}\{\mathsf{Energy}_\alpha(\mathbb{O})\} < +\infty$ for all $0 < \alpha < 2$. But this is easy. Indeed, note that

$$\mathbb{E}\Big\{\mathsf{Energy}_\alpha(\mathbb{O})\Big\} = 2\int_1^2 \int_s^2 \mathbb{E}\big\{|B_{t-s}|^{-\alpha}\big\}\, ds\, dt = 2\int_1^2 \int_s^2 |t-s|^{-\frac{\alpha}{2}}\, ds\, dt \times \mathbb{E}\{|Z|^{-\alpha}\},$$

where Z is a d-dimensional vector of i.i.d. standard normals. Since $\alpha < 2$, the double integral is finite. It suffices to show that $\mathbb{E}\{|Z|^{-\alpha}\rangle < +\infty$. But

$$\begin{aligned}
\mathbb{E}\{|Z|^{-\alpha}\} &= \int_0^\infty \mathbb{P}\{|Z|^{-\alpha} > \lambda\}\, d\lambda \\
&\leq 1 + \int_1^\infty \mathbb{P}\{|Z|^{-\alpha} > \lambda\}\, d\lambda \\
&= 1 + \alpha \int_0^1 \mathbb{P}\{|Z| < u\} u^{-\alpha-1}\, du && (u = \lambda^{-\frac{1}{\alpha}}) \\
&= 1 + \alpha \int_0^1 \big[\mathbb{P}\{|Z_1| \leq u\}\big]^d u^{-\alpha-1}\, du.
\end{aligned}$$

But $\mathbb{P}\{|Z_1| \leq u\} = (2\pi)^{-\frac{1}{2}} \int_{-u}^u e^{-\frac{1}{2}\lambda^2}\, d\lambda \leq u$. Hence, using $d \geq 2 > \alpha$,

$$\mathbb{E}\{|Z|^{-\alpha}\} \leq 1 + \alpha \int_0^1 u^{d-\alpha-1}\, du = \frac{d}{d-\alpha} < +\infty,$$

as promised. □

Here is a slick proof of Lévy's theorem alluded to earlier.

Theorem 3.2 (P. Lévy) *If B is Brownian motion in $\mathbb{R}^d$ and if $d \geq 2$, $B(\mathbb{R}_+)$ has zero Lebesgue's measure, a.s.*

Proof I will prove this when $d \geq 3$ where things are alot simpler, and appeal to an argument that, in physics literature, is called *group renormalization*. Tacitly held, throughout, is the fact that $\mathbb{E}\{\lambda_d(B(0,t))\} < \infty$; this is a ready consequence of the easy estimate $\mathbb{E}\{\sup_{0\leq s\leq t}|B_s|^d\} < +\infty$.

If λ_d denotes Lebesgue's measure on $\mathbb{R}^d$, note that

$$\mathbb{E}\{\lambda_d(B(0,2))\} \leq \mathbb{E}\{\lambda_d(B(0,1))\} + \mathbb{E}\{\lambda_d(B(1,2))\}.$$

We make two observations: (i) $\mathbb{E}\{\lambda_d(B(1,2))\} = \mathbb{E}\{\lambda_d(B(0,1))\}$; and (ii) by Brownian scaling, $\mathbb{E}\{\lambda_d(B(0,2))\} = \mathbb{E}\{\lambda_d(\sqrt{2}B(0,1))\} = 2^{\frac{d}{2}}\mathbb{E}\{\lambda_d(B(0,1))\}$, thanks to the scaling properties of λ_d. Combining these observations, we get $2^{\frac{d}{2}}\mathbb{E}\{\lambda_d(B(0,1))\} \leq 2\mathbb{E}\{\lambda_d(B(0,1))\}$, which is impossible unless $\mathbb{E}\{\lambda_d(B(0,1))\} = 0$, since $d \geq 3$.

A few words about the $d = 2$ case: Lévy's theorem is harder to prove when $d = 2$, and uses the estimate (3.1) and a covering argument. □

4 Brownian Motion and Newtonian Capacity

We now look at an elementary connection between three-dimensional Brownian motion and Newtonian capacity. Let $B = \{B_t;\, t \geq 0\}$ denote three-dimensional Brownian motion, and consider the linear operator

$$\mathcal{U}f(x) = \mathbb{E}\Big\{ \int_0^\infty f(B_s + x)\, ds \Big\}.$$

Here, $x \in \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}_+$ is measurable. We can easily evaluate this as follows. A few liberal doses of Fubini-Tonelli yield:

$$\begin{aligned}\mathcal{U}f(x) &= \int_0^\infty \int_{\mathbb{R}^3} f(x+z) \frac{e^{-\frac{\|z\|^2}{2s}}}{(2\pi s)^{\frac{3}{2}}}\, dz\, ds \\ &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \|x-y\|^{-1} f(y)\, dy.\end{aligned}$$

Now, suppose f is a probability density function on some (say, nice compact) set $E \subset \mathbb{R}^3$. Then, $\mathcal{U}f(x)$ is the expected amount of time spent in E, *weighed according to* f, and starting at $x \in \mathbb{R}^3$. Now, suppose the Brownian motion itself starts according to the pdf f. Then, this expected time is

$$\int_{\mathbb{R}^3} \mathcal{U}f(x) f(x)\, dx = \frac{1}{4\pi} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \|x-y\|^{-1} f(y)\, dy\, f(x)\, dx.$$

You should recognize the right hand side as $(4\pi)^{-1}$ times the Newtonian energy of the measure $f(x)\,dx$. In summary, if we start Brownian motion in E according to f, the expected amount of time spent in E, weighed accroding to f, if precisely $\frac{1}{4\pi}\mathsf{Energy}_1(f)$, where $f(x)$ is identified with the measure $f(x)dx$ here.

SOMETHING TO TRY: Check that whenever B is Brownian motion in $\mathbb{R}^d$ ($d \geq 3$) that starts according to some pdf f,

$$\int_{\mathbb{R}^d} \mathbb{E}\Big\{ \int_0^\infty f(B_s + x)\, ds \Big\} f(x)\, dx = c\mathsf{Energy}_{d-2}(f),$$

and compute c. Why does this fail when $d = 2$ or $d = 1$?

5 Riesz Transforms and $\mathcal{H}_p$ Spaces

Below, if will prove that if $\alpha \in (0, d)$, the Fourier transform of the function $\mathbb{R}^d \ni \xi \mapsto \|\xi\|^{-\alpha}$ is a constant multiple of $\|\xi\|^{-(d-\alpha)}$. We will compute this constant also. However, let us see what this implies for energy. Recall that

$$\mathsf{Energy}_\alpha(\mu) = \int_{\mathbb{R}^d} \mathcal{R}\mu(x)\, \mu(dx),$$

where $\mathcal{R}\mu(x) = \int_{\mathbb{R}^d} |x-y|^{-\alpha}\, \mu(dy)$. We are only interested in whether or not the above is finite for some measure μ. Thus, we cheat at the last moment and replace the ℓ^∞ norm by ℓ^2 norm to get $\mathsf{Energy}_\alpha(\mu) = \int_{\mathbb{R}^d} \mathcal{R}\mu(x)\, \mu(dx)$, where

$$\mathcal{R}\mu(x) = \int_{\mathbb{R}^d} \|x-y\|^{-\alpha}\, \mu(dy)$$

is the so-called α-dimensional *Riesz transform* of μ. Using obvious L^2 notation, $\mathsf{Energy}_\alpha(\mu) = (\mathcal{R}\mu, \mu)$. Having noted the L^2 connection, we apply Fourier transforms (via Plancherel) to see that

$$\mathsf{Energy}_\alpha(\mu) = (2\pi)^{-d}(\widehat{\mathcal{R}\mu}, \widehat{\mu}).$$

But, $\mathcal{R}$ is a convolution operator that can be identified with the kernel $\mathcal{R}(a) = \|a\|^{-\alpha}$. Therefore, $\widehat{\mathcal{R}\mu} = \widehat{\mathcal{R}}\widehat{\mu}$. On the other hand, we just mentioned that the Fourier transform of $\mathcal{R}$ is $c\|\xi\|^{-(d-\alpha)}$, where $c = C_\alpha$ from Lemma 5.1 below. Thus, when $\alpha \in (0, d)$,

$$\mathsf{Energy}_\alpha(\mu) = (2\pi)^{-d} c \int_{\mathbb{R}^d} \|\xi\|^{-(d-\alpha)} |\widehat{\mu}(\xi)|^2\, d\xi.$$

Thus, $\mathsf{Energy}_\alpha(\mu)$ is equivalent (i.e., converges iff the following does) to the $\mathcal{H}_{\frac{d-\alpha}{2}}$-norm:

$$\|\mu\|^2_{\frac{d-\alpha}{2}} = \int_{\mathbb{R}^d} \big|\widehat{\mu}(\xi)\big|^2 \big\{1 + \|\xi\|^2\big\}^{-\frac{d-\alpha}{2}}\, d\xi.$$

Thus we have linked the computations of the lecture of D. BLOUNT earlier this week to energy computations (in his notation, $\gamma = -\frac{1}{2}(d-\alpha)$.) If you want to have a more in-depth look at connections to energy and dimension, try the following.

SOMETHING TO TRY: Let $X = \{X_t;\, t \geq 0\}$ be an isotropic Lévy process in $\mathbb{R}^d$. That is, a process with i.i.d. increments whose characteristic function is given by $\mathbb{E}\{e^{i\xi\cdot X_t}\} = \exp\{-\frac{t}{2}\|\xi\|^\alpha\}$. It is known that $\alpha \in (0, 2]$ is necessary. When $\alpha = 2$, X is just Brownian motion.

Suppose further that $\alpha \in (0, d)$, and define the weighted occupation measure

$$\mathbb{O}(E) = \int_0^\infty \mathbf{1}_E(X_s) e^{-s}\, ds.$$

(i) Check that its Fourier transform is $\widehat{\mathbb{O}}(\xi) = \int_0^\infty e^{i\xi\cdot X_s} e^{-s}\, ds$.

(ii) Use the above to show that $\mathbb{E}\{|\widehat{\mathbb{O}}(\xi)|^2\} = \left\{\frac{1}{2}\|\xi\|^\alpha + 1\right\}^{-1}$. In particular, for any $\beta \in (0, d)$,

$$\mathbb{E}\big\{\mathsf{Energy}_\beta(\mathbb{O})\big\} = (2\pi)^{-d} c \int_{\mathbb{R}^d} \|\xi\|^{-(d-\beta)} \left\{\tfrac{1}{2}\|\xi\|^\alpha + 1\right\}^{-1} d\xi.$$

(iii) Conclude that whenever $\beta < \alpha$, $\mathsf{Energy}_\beta(\mathbb{O}) < +\infty$, a.s.

(iv) Use Frostman's theorem to show that, with probability one, $\dim(X(\mathbb{R}_+)) \geq \alpha$. One can show that this is sharp. That is, when $\alpha \in (0, d)$, $\dim(X(\mathbb{R}_+)) = \alpha$, a.s. When $\alpha = 2$, we did this last part explicitly in Theorem 3.1. Here, the strategy is the same, but we need hitting probability estimates for stable processes.

Let me conclude with the following promised calculation, then. Henceforth, for any $\gamma > 0$,

$$\Psi_\gamma(\xi) = \|\xi\|^{-\gamma}.$$

Lemma 5.1 *For any $\gamma \in (0, d)$, $\widehat{\Psi_\gamma} = \frac{1}{C_\gamma}\Psi_{d-\gamma}$, where $C_\gamma = 2^{-\frac{\gamma}{2}}\pi^{-\frac{d}{2}}\Gamma(\frac{d-\gamma}{2})/\Gamma(\frac{\gamma}{2})$.*

Proof We will relate the mention Fourier transform to the Laplace transform of a Gaussian, first. This may seem like magic, but if you apply some more Fourier analysis (namely, Bochner's subordination), you can explain this more clearly; cf. MPP for the latter.

Note that for $\theta > 0$ and $\beta > -1$, $\int_0^\infty e^{-t\theta} t^\beta\, dt = \theta^{-(1+\beta)}\Gamma(1+\beta)$. In particular,

$$\int_0^\infty e^{-t\|\xi\|^2} t^\beta\, dt = \|\xi\|^{-(2+2\beta)}\Gamma(1+\beta).$$

Now, take a well-tempered function φ and consider

$$\int_{\mathbb{R}^d} \overline{\widehat{\varphi}(\xi)} \|\xi\|^{-(2+2\beta)}\, d\xi = \frac{1}{\Gamma(1+\beta)} \int_0^\infty t^\beta \Big(\int_{\mathbb{R}^d} \overline{\widehat{\varphi}(\xi)} e^{-t\|\xi\|^2}\, d\xi \Big)\, dt.$$

(In truth, the φ should be a tempered distribution.) Apply Parseval's identity: $(\widehat{\varphi}, \widehat{f}) = (2\pi)^{-d}(\varphi, f)$ to deduce

$$\begin{aligned}
\int_{\mathbb{R}^d} \overline{\widehat{\varphi}(\xi)} \|\xi\|^{-(2+2\beta)}\, d\xi &= \frac{(2\pi)^d}{\Gamma(1+\beta)} \int_0^\infty t^\beta \Big(\int_{\mathbb{R}^d} \varphi(\xi) \frac{e^{-\|\xi\|^2/2t}}{(2\pi t)^{\frac{d}{2}}}\, d\xi \Big)\, dt \\
&= \frac{(2\pi)^d}{\Gamma(1+\beta)} \int_{\mathbb{R}^d} \varphi(\xi) \Big(\int_0^\infty \frac{e^{-\|\xi\|^2/2t}}{(2\pi t)^{\frac{d}{2}}} t^\beta\, dt \Big)\, d\xi \qquad (s = \tfrac{\|\xi\|^2}{2t}) \\
&= \frac{(2\pi)^d}{\Gamma(1+\beta)} (2\pi)^{-\frac{d}{2}} 2^{\frac{d}{2}-\beta-1} \Gamma(\tfrac{d}{2} - \beta - 1) \int_{\mathbb{R}^d} \varphi(\xi) \|\xi\|^{d-(2\beta+2)}\, d\xi.
\end{aligned}$$

Let $2\beta + 2 = \gamma$ to finish. $\square$

Lecture 5

Brownian Sheet, Potential Theory, and Kahane's Problem

Recall that a Brownian sheet $B = \{B(s,t);\ s,t \geq 0\}$ is just a real process defined as

$$B(s,t) = \mathbb{W}([0,s] \times [0,t]), \qquad \forall s,t \geq 0,$$

where $\mathbb{W}$ denotes white noise on $\mathbb{R}^2$. We will refer to this as *one-dimensional Brownian sheet* to stress that the process takes its values in $\mathbb{R}$.

By a *d-dimensional Brownian sheet*, we mean the d-dimensional process $B = \{B(s,t);\ s,t \geq 0\}$ such that $B_1, B_2, \ldots, B_d$ are i.i.d. (1-dimensional) Brownian sheets. Of course, $B_i = \{B_i(s,t);\ s,t \geq 0\}$.

1 Polar Sets

The following theorem, due to S. Kakutani, is the cornerstone of probabilistic potential theory.

Theorem 1.1 (S. Kakutani) *Let b denote d-dimensional Brownian motion, and consider a fixed compact $E \subset \mathbb{R}^d$. Then,*

$$\mathbb{P}\{\exists t > 0 :\ b_t \in E\} > 0 \iff \mathsf{Cap}_{d-2}(E) > 0.$$

The above relates E to what is called a *polar set*. Probabilistically, a set E is polar for a process $X = \{X_t;\ t \in T\}$ if with positive probability, $\exists t \in T :\ X_t \in E$. Thus, Kakutani's theorem characterizes polar sets for Brownian motion. This notion of polarity matches with the one from harmonic analysis, which has to do with the removable singularities of the Dirichlet problem off the set. Amongst other things, fairly routine calculations show that the α-dimensional capacity of a ball of radius ε is of order ε^α if $\alpha > 0$, and $[\log(1/\varepsilon)]^{-1}$ if $\alpha = 0$. That is, Theorem 1.1 contains the hitting probability estimate (3.1) of Lecture 4.

A more recent result, this time for the sheet is,

Theorem 1.2 (D. Kh. and Z. Shi) *If E is a compact set in $\mathbb{R}^d$, $\mathbb{P}\{B(\mathbb{R}^2_+) \cap E \neq \varnothing\} > 0$ iff $\mathsf{Cap}_{d-4}(E) > 0$. In fact, for any $M > 0$, there exists c_1 and c_2, such that for all compact $E \subset [-M, M]^d$,*

$$c_1 \mathsf{Cap}_{d-4}(E) \leq \mathbb{P}\{B[1,2]^2 \cap E \neq \varnothing\} \leq c_2 \mathsf{Cap}_{d-4}(E).$$

We will prove this shortly. However, let me mention a variant: suppose $X_1, \ldots, X_N$ are i.i.d. isotropic stable processes in $\mathbb{R}^d$ all with index $\alpha \in (0, 2]$. This means that each X_ℓ is an $\mathbb{R}^d$-valued Lévy process with characteristic function

$$\mathbb{E}\{e^{i\xi \cdot X_\ell(t)}\} = \exp\big(-\tfrac{1}{2}\|\xi\|^\alpha\big), \qquad \forall \xi \in \mathbb{R}^d,\ t \geq 0,\ \ell = 1, \ldots, d.$$

The $\frac{1}{2}$ is to ensure that when $\alpha = 2$, X_ℓ is standard Brownian motion. The α-dimensional, N-parameter *additive stable process* is the random field

$$X(\mathbf{t}) = X_1(t_1) + \cdots + X_N(t_N), \qquad \forall \mathbf{t} \in \mathbb{R}^N_+.$$

Theorem 1.3 (F. Hirsch and S. Song; MPP Ch. 11) *If X is an N-parameter, d-dimensional additive stable process of index α, and if $E \subset \mathbb{R}^d$ is a given compact set, then E is polar for X iff $\mathsf{Cap}_{d-\alpha N}(E) > 0$.*

The above, together with the energy/covering arguments of Lecture 4 (cf. Theorem 3.1 there), this shows

Theorem 1.4 *If X is an N-parameter, d-dimensional additive stable process of index α,*

$$\dim(X(\mathbb{R}^N_+)) = \alpha N \wedge d, \qquad \textit{a.s.}$$

The above two theorems provide us with processes that correspond to arbitrary dimensions and capacities.

2 Application to Stochastic Codimension

The preceeding has a remarkable consequence about a large class of random sets. We say that a random set $X \subset \mathbb{R}^d$ has *codimension* β, if β is the critical number such that for all compact sets $E \subset \mathbb{R}^d$

with $\dim(E) > \beta$, $\mathbb{P}\{X \cap E \neq \varnothing\} > 0$, while for all compact sets $F \subset \mathbb{R}^d$ with $\dim(F) < \beta$, $\mathbb{P}\{X \cap F \neq \varnothing\} = 0$. The notion of codimension was coined in this way in Kh-Shi '99, but the essential idea has been around in the works of Taylor '65, Lyons '99, Peres '95, $\cdots$

When it does exist, the codimension of a random set is a nonrandom number.

Warning: Not all random sets have a codimension.

As examples of random sets that *do* have codimension, we mention the following consequence of Theorem 1.3:

Corollary 2.1 *If Z denotes an (N, d)-additive stable process of index $\alpha \in (0, 2]$,* $\text{codim}(Z[1,2]^N) = d - \alpha N$.

We now wish to use Theorem 1.3 to prove the following result. In the present form, it is from MPP Ch. 11, but for $d = 1$, it is from Kh-Shi '99.

Theorem 2.2 (MPP Ch. 11) *If X is a random set in $\mathbb{R}^d$ that has codimension $\beta \in (0, d)$,*

$$\dim(X) = d - \text{codim}(X), \qquad \textit{a.s.}$$

That is, in the best of circumstances,

$$\boxed{\dim(X) + \text{codim}(X) = \text{topological dimension.}}$$

A note of warning: if X is not compact, $\dim(X)$ can be defined by $\sup_{n \geq 1} \dim(\overline{X \cap [-n, n]^d})$.

The proof depends on the following result that can be found in the works of Yuval Peres '95, but with percolation proofs.

Lemma 2.3 (Peres' lemma) *For each $\beta \in (0, d)$, there exists a random set Λ_β, whose codimension is β. Moreover, $\dim(\Lambda_\beta) = d - \beta$, almost surely.*

Proof Let $\Lambda_\beta = Z(\mathbb{R}^N_+)$, where Z is an (N,d)-addtive stable process. The result follows from Corollary 2.1 and Theorem 1.4. $\square$

Proof of Theorem 2.2 By localization, we may assume that X is a.s. compact. Let $\Lambda_\beta = \cup_{i=1}^\infty \Lambda_\beta^i$, where $\Lambda_\beta^1, \Lambda_\beta^2, \ldots$ are iid copies of the sets in Peres' lemma, and are all totally independent of our random set X. Then, by Peres' lemma and by the lemma of Borel–Cantelli,

$$\mathbb{P}\{\Lambda_\beta \cap X \neq \varnothing \,|\, X\} = \begin{cases} 0, & \text{on } \{\dim(X) < \beta\} \\ 1, & \text{on } \{\dim(X) > \beta\} \end{cases}.$$

On the other hand, by the very definition of codimension,

$$\mathbb{P}\{\Lambda_\beta \cap X \neq \varnothing \,|\, \Lambda_\beta\} = \begin{cases} 0, & \text{if } \operatorname{codim}(X) > d - \beta = \dim(\Lambda_\beta) \\ > 0, & \text{if } \operatorname{codim}(X) < d - \beta \end{cases}.$$

Take expectations of the last two displays to see that for any $\beta \in (0,d)$,

$$\begin{aligned} \operatorname{codim}(X) < d - \beta &\implies \dim(X) \geq \beta, \text{ a.s.} \\ \operatorname{codim}(X) > d - \beta &\implies \dim(X) \leq \beta, \text{ a.s.} \end{aligned}$$

This easily proves our theorem. $\square$

3 Proof of Theorem 1.2

We begin with an elementary, though extremely useful, lemma.

Lemma 3.1 (R. E. A. C. Paley and A. Zygmund) *If $Z \geq 0$ a.s., and if $Z \in L^2(\mathbb{P})$,*

$$\mathbb{P}\{Z > 0\} \geq \frac{|\mathbb{E}\{Z\}|^2}{\mathbb{E}\{Z^2\}},$$

where $0 \div 0 = 0$.

Proof By the Cauchy–Schwarz inequality,

$$\begin{aligned} \mathbb{E}\{Z\} &= \mathbb{E}\{Z;\ Z > 0\} \\ &\leq \sqrt{\mathbb{E}\{Z^2\}\mathbb{P}\{Z > 0\}}. \end{aligned}$$

Square and solve. $\square$

Roughly speaking, the strategy of proof of Theorem 1.2 is to show that $B[1,2]^2$ intersects E iff the occupation measure evaluated at some $f \in \mathcal{P}(E)$ is large, where $f \in \mathcal{P}(E)$ means that f is a probability density function supported on E, where the latter is given by

$$\mathbb{O}(f) = \int_{[1,2]^2} f(B(s,t))\, ds\, dt.$$

Lemma 3.2 *For each $M > 0$, there exists a constant $c = c(M) > 1$ such that for all pdf's f on $[-M, M]^d$,*

$$\mathbb{E}\{\mathbb{O}(f)\} \geq c.$$

Proof The law of the variate $B(s,t)$ is $\sqrt{st}Z$, where $Z = (Z_1, \ldots, Z_d)$ is a vector of i.i.d. standard normals. The lemma follows from direct compmutations, since this pdf is easily seen to be bounded below on $[-M, M]^d$. $\square$

Lemma 3.3 *For each $M > 0$, there exists a constant $c = c(M) > 1$ such that for all pdf's f on $[-M, M]^d$,*

$$\mathbb{E}\left\{ \left|\mathbb{O}(f)\right|^2 \right\} \leq c\mathsf{Energy}_{d-4}(f).$$

To develop Kakutani's theorem by the methods of this lecture, start with

SOMETHING TO TRY: Let $b = \{b_t;\, t \geq 0\}$ denote d-dimensional Brownian motion. Then, show that for each $M > 0$, there exists a constant $c = c(M) > 1$ such that for all pdf's f on $[-M, M]^d$,

$$\mathbb{E}\left\{ \left| \int_1^2 f(b_s)\, ds \right|^2 \right\} \leq c\mathsf{Energy}_{d-2}(f).$$

Also show that there exists $c = c(M) > 0$, such that for all pdf's f on $[-M, M]^d$,

$$\mathbb{E}\left\{ \int_1^2 f(b_s)\, ds \right\} \geq c.$$

Proof of Lemma 3.3 Note that

$$\begin{aligned}\mathbb{E}\left\{\left|\mathbb{O}(f)\right|^2\right\} &= \mathbb{E}\Big\{\int_{[1,2]^2}\int_{[1,2]^2} f(B(\mathbf{s}))f(B(\mathbf{t}))\,d\mathbf{s}\,d\mathbf{t}\Big\}\\ &= \mathbb{E}\Big\{\int_{[1,2]^2}\int_{[1,2]^2} f\big(B(\mathbf{s}\curlywedge\mathbf{t})+\xi_1\big)f\big(B(\mathbf{s}\curlywedge\mathbf{t})+\xi_1\big)\,d\mathbf{s}\,d\mathbf{t}\Big\}\end{aligned}$$

where

$$\begin{aligned}\xi_1 &= B(\mathbf{s})-B(\mathbf{s}\curlywedge\mathbf{t}),\ \text{and}\\ \xi_2 &= B(\mathbf{t})-B(\mathbf{s}\curlywedge\mathbf{t}).\end{aligned}$$

Note that (i) ξ_1 and ξ_2 are independent; and (ii) the pdf of $B(\mathbf{s}\curlywedge\mathbf{t})$ is bounded above by a constant, uniformly for all $\mathbf{s},\mathbf{t}\in[1,2]^2$. Indeed, the latter pdf, at $x\in\mathbb{R}^d$, is

$$(2\pi)^{-\frac{d}{2}}(s_1\wedge t_1)^{-\frac{d}{2}}(s_2\wedge t_2)^{-\frac{d}{2}}\exp\Big(-\frac{\|x\|^2}{2(s_1\wedge t_1)(s_2\wedge t_2)}\Big)\leq 1.$$

Thus,

$$\begin{aligned}\mathbb{E}\left\{\left|\mathbb{O}(f)\right|^2\right\} &\leq \mathbb{E}\Big\{\int_{[1,2]^2}\int_{[1,2]^2}\int_{\mathbb{R}^d} f(x+\xi_1)f(x+\xi_2)\,dx\,d\mathbf{s}\,d\mathbf{t}\Big\}\\ &= \mathbb{E}\Big\{\int_{[1,2]^2}\int_{[1,2]^2}\int_{\mathbb{R}^d} f(x)f(x+\xi_2-\xi_1)\,dx\,d\mathbf{s}\,d\mathbf{t}\Big\}\\ &= \mathbb{E}\Big\{\int_{[1,2]^2}\int_{[1,2]^2}\int_{\mathbb{R}^d} f(x)f\big(x+B(\mathbf{t})-B(\mathbf{s})\big)\,dx\,d\mathbf{s}\,d\mathbf{t}\Big\}.\end{aligned}$$

Now, we proceed to a variance estimate as we did in Lemma 2.2 of Lecture 2 for $d=1$. Indeed, the variance of each of the coordinates of $B(\mathbf{t})-B(\mathbf{s})$ is bounded above and below by constant multiples of $|\mathbf{t}-\mathbf{s}|$. This leads to

$$\begin{aligned}\mathbb{E}\left\{\left|\mathbb{O}(f)\right|^2\right\} &\leq c\int_{[1,2]^2}\int_{[1,2]^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)f(x+y)\frac{e^{-\frac{|y|^2}{c|\mathbf{t}-\mathbf{s}|}}}{|\mathbf{t}-\mathbf{s}|^{\frac{d}{2}}}\,dx\,dy\,d\mathbf{s}\,d\mathbf{t}\\ &= c\mathsf{Energy}_{d-4}(f),\end{aligned}$$

after a few more lines of calculations. □

Proof of Theorem 1.2: Lower Bound If there are pdf's supported by E, choose f to be any one of them and note that

$$\mathbb{O}(f)>0 \implies B[1,2]^2\cap E\neq\varnothing.$$

So, combine this with Lemmas 3.2 and 3.3 to get

$$\begin{aligned}\mathbb{P}\{B[1,2]^2 \cap E \neq \varnothing\} &\geq \mathbb{P}\{\mathbb{O}(f) > 0\} \\ &\geq \frac{|\mathbb{E}\{\mathbb{O}(f)\}|^2}{\mathbb{E}\{|\mathbb{O}(f)|^2\}} \\ &\geq c\big[\mathsf{Energy}_{d-4}(f)\big]^{-1}.\end{aligned}$$

We have used the Paley–Zygmund inequality in the second-to-last line; cf. Lemma 3.1. This holds uniformly over all pdf's f on E. Replace E by its close ε-enlargement E_ε, we get

$$\mathbb{P}\{B[1,2]^2 \cap E_\varepsilon \neq \varnothing\} \geq c\Big[\inf_{\substack{\mu\in\mathcal{P}(E_\varepsilon):\\ \mu=\text{absolutely continuous}}} \mathsf{Energy}_{d-4}(\mu)\Big]^{-1}.$$

Any $\mu \in \mathcal{P}(E)$ can be approximated, in the sense of weak convergence, by absolutely continuous $\mu_\varepsilon \in \mathcal{P}(E_\varepsilon)$. A little Fourier analysis, then, shows that as $\varepsilon \to 0$, $\mathsf{Energy}_\alpha(\mu)$ is approximable by $\mathsf{Energy}_\alpha(\mu_\varepsilon)$; cf. the last section of Lecture 4 for the requisite material on Fourier analysis. On the other hand, $\mathbb{P}\{B[1,2]^2 \cap E_\varepsilon \neq \varnothing\} \to \mathbb{P}\{B[1,2]^2 \cap E \neq \varnothing\}$, as $\varepsilon \to 0$. This yields,

$$\mathbb{P}\{B[1,2]^2 \cap E \neq \varnothing\} \geq c\Big[\inf_{\mu\in\mathcal{P}(E)} \mathsf{Energy}_{d-4}(\mu)\Big]^{-1},$$

which equals $c\mathsf{Cap}_{d-4}(E)$. □

We now turn to the more difficult

Proof of Theorem 1.2: Upper Bound (sketch) Define the 2-parameter martingale Mf by

$$Mf(\mathbf{t}) = \mathbb{E}\{\mathbb{O}(f) \,|\, \mathcal{F}(\mathbf{t})\},$$

where f is a pdf on E_ε, and $\mathcal{F}$ is the natural 2-parameter filtration of B. Clearly, for any $\mathbf{s} \in [1, \frac{3}{2}]^2$,

$$\begin{aligned}Mf(\mathbf{s}) &= \int_{[1,2]^2} \mathbb{E}\big\{ f(B(\mathbf{t})) \,\big|\, \mathcal{F}(\mathbf{s})\big\}\, d\mathbf{t} \\ &\geq \int_{\substack{\mathbf{t}\succcurlyeq\mathbf{s}:\\ \mathbf{t}\in[1,2]^N}} \mathbb{E}\big\{ f(B(\mathbf{t})) \,\big|\, \mathcal{F}(\mathbf{s})\big\}\, d\mathbf{t} \\ &= \int_{\substack{\mathbf{t}\succcurlyeq\mathbf{s}:\\ \mathbf{t}\in[1,2]^N}} \mathbb{E}\big\{ f\big(B(\mathbf{t}) - B(\mathbf{s}) + B(\mathbf{s})\big) \,\big|\, \mathcal{F}(\mathbf{s})\big\}\, d\mathbf{t}.\end{aligned}$$

Now, recall that whenever $\mathbf{t} \succcurlyeq \mathbf{s}$, $B(\mathbf{t}) - B(\mathbf{s})$ is independent of $\mathcal{F}(\mathbf{s})$, and whose coordinatewise variance is, upto a constant, $|\mathbf{t}-\mathbf{s}|$. A few more lines show that for any $\mathbf{s} \in [1, \frac{3}{2}]^2$,

$$Mf(\mathbf{s}) \geq c\mathfrak{G}f\big(B(\mathbf{s})\big),$$

where $\mathfrak{G}\mu(x) = \int_{\mathbb{R}^d} |x-y|^{-d+4}\, \mu(dy)$ if $d > 4$, $\mathfrak{G}\mu(x) = \int_{\mathbb{R}^d} \log_+(\frac{1}{|x-y|})\, \mu(dy)$, if $d = 4$, and $\mathfrak{G}\mu(x) = 1$, if $d < 4$. Let $\mathbf{T}$ be any measurable variate in $[1, \frac{3}{2}]^2 \cup \{\infty\}$, such that $\mathbf{T} \neq \infty$ iff $\exists \mathbf{s} \in [1, \frac{3}{2}]^2$ such that $B(\mathbf{s}) \in E$, and in which case, $B(\mathbf{T}) \in E$. Then, ignoring null sets, we have

$$\sup_{\mathbf{s}\in[1,\frac{3}{2}]^2} Mf(\mathbf{T}) \geq c\mathfrak{G}f\big(B(\mathbf{T})\big) \cdot \mathbf{1}_{\{\mathbf{T}\neq\infty\}}.$$

Square both sides, and take expectations:

$$\mathbb{E}\Big\{ \sup_{\mathbf{s}\in[1,\frac{3}{2}]^2} \big|Mf(\mathbf{T})\big|^2 \Big\} \geq c\mathbb{E}\Big\{ \big|\mathfrak{G}f\big(B(\mathbf{T})\big)\big|^2 \cdot \mathbf{1}_{\{\mathbf{T}\neq\infty\}} \Big\}.$$

Now, if $\mathbb{P}\{\mathbf{T} \neq \infty\} = 0$, there is nothing to prove. Else, choose $\mu(\bullet) = \mathbb{P}\{B(\mathbf{T}) \in \bullet \,|\, \mathbf{T} \neq \infty\}$ ($\mu \in \mathcal{P}(E)$), to see that

$$\begin{aligned}
\mathbb{E}\Big\{ \sup_{\mathbf{s}\in[1,\frac{3}{2}]^2} \big|Mf(\mathbf{T})\big|^2 \Big\} &\geq c\mathbb{E}\Big\{ \big|\mathfrak{G}f\big(B(\mathbf{T})\big)\big|^2 \,\Big|\, \mathbf{T} \neq \infty \Big\} \times \mathbb{P}\{\mathbf{T} \neq \infty\} \\
&\geq c\Big|\mathbb{E}\Big\{ \mathfrak{G}f\big(B(\mathbf{T})\big) \,\Big|\, \mathbf{T} \neq \infty \Big\}\Big|^2 \times \mathbb{P}\{\mathbf{T} \neq \infty\} \\
&= c\Big| \int \mathfrak{G}f(x)\, \mu(dx) \Big|^2 \times \mathbb{P}\{\mathbf{T} \neq \infty\}.
\end{aligned}$$

On the other hand, by Cairoli's inequality, the left hand side is bounded above by $16\mathbb{E}\{|\mathbb{O}(f)|^2\} \leq c'\mathsf{Energy}_{d-4}(f)$, thanks to Lemma 3.3. Combining things, we have, for this special $\mu \in \mathcal{P}(E)$,

$$c\mathsf{Energy}_{d-4}(f) \geq \Big| \int \mathfrak{G}f(x)\, \mu(dx) \Big|^2 \times \mathbb{P}\{B[1, \tfrac{3}{2}]^2 \cap E \neq \varnothing\}.$$

This holds for any pdf f. Now, choose pdf's f that converge weakly to μ. A little Fourier analysis shows that we can do this so that the energies of the f's also approximate that of μ, and $\int \mathfrak{G}f(x)\, \mu(dx) \simeq \int \mathfrak{G}\mu(x)\, \mu(dx) = \mathsf{Energy}_{d-4}(\mu)$. This completes our proof. $\square$

SOMETHING TO TRY: Try and mimick the above sketched proof to show the upper bound in Kakutani's theorem. You may ignore the Fourier analysis details.

4 Kahane's Problem

We come to the last portion of these lectures, which is on a class of problems that I call Kahane's problem, due to the work of J.-P. Kahane in this area.

Kahane's problem for a random field X is: "*when does $X(E)$ have positive Lebesgue's measure?*" I will work the details out for Brownian motion, where things are easier. The problem for the Brownian sheet was partly solved by Kahane (cf. his '86 book) and completely solved by Kh. '99 in case $N = 2$. Recent work of Kh. and Xiao '01 has completed the solution to Kahane's problem and a class of related problems, and we hope to write this up at some point. Here is the story for Brownian motion, where we work things out more or less completely. The story for Brownian sheet is more difficult, and I will say some words about the details later.

Theorem 4.1 (Kahane; Hawkes) *If B denotes Brownian motion in $\mathbb{R}$, and if $E \subset \mathbb{R}_+$ is compact, then*

$$\mathbb{E}\{|B(E)|\} > 0 \iff \mathsf{Cap}_{\frac{1}{2}}(E) > 0.$$

In particular,

$$\begin{aligned} \dim(E) > \tfrac{1}{2} &\implies |B(E)| > 0, \text{ with positive probability} \\ \dim(E) < \tfrac{1}{2} &\implies |B(E)| = 0, \text{ a.s.} \end{aligned}$$

You can interpret this as a statement about hitting proabilities for the level sets of Brownian motion, viz.,

$$\int_{\mathbb{R}^d} \mathbb{P}\{B^{-1}\{a\} \cap E \neq \varnothing\}\, da > 0 \iff \mathsf{Cap}_{\frac{1}{2}}(E) > 0.$$

I will prove the following for Brownian motion. It clearly implies the above theorem upon integration.

Theorem 4.2 *Suppose $E \subset [1, 2]$ is compact, and fix $M > 0$. Then, there exists c_1 and c_2 such that for all $|a| \leq M$,*

$$c_1 \mathsf{Cap}_{\frac{1}{2}}(E) \leq \mathbb{P}\{a \in B(E)\} \leq c_2 \mathsf{Cap}_{\frac{1}{2}}(E).$$

As a simple consequence of this and Frostman's theorem, we see that the critical dimension for $B^{-1}\{0\}$ to hit a set is $\frac{1}{2}$. Equivalently, the zero set, $B^{-1}\{0\}$, has codimension $\frac{1}{2}$. Since the topological dimension of $B^{-1}\{0\}$ is 1, by Theorem 2.2,

Corollary 4.3 (P. Lévy) *With probability one,* $\dim B^{-1}\{0\} = \frac{1}{2}$.

Proof Without loss of any generality, we may and will assume that $E \subseteq [0,1]$.

For any $\mu \in \mathcal{P}(E)$ and for all $a \in \mathbb{R}$, define

$$J_\varepsilon^a(\mu) = (2\varepsilon)^{-1} \int_0^\infty \mathbf{1}_{\{|B_s - a| \le \varepsilon\}} \, \mu(ds).$$

Then, for every $M > 0$, there exists c such that

$$\begin{aligned} \inf_{\varepsilon \in (0,1)} \inf_{a \in [-M,M]} \mathbb{E}\{J_\varepsilon^a(\mu)\} \ge c, \text{ and} \\ \sup_{a \in \mathbb{R}} \sup_{\varepsilon \in (0,1)} \mathbb{E}\{|J_\varepsilon^a(\mu)|^2\} \le \mathsf{Energy}_{\frac{1}{2}}(\mu). \end{aligned} \tag{4.1}$$

Now, we apply Paley–Zygmund inequality:

$$\begin{aligned} \mathbb{P}\{a \in B(E)\} &\ge \mathbb{P}\{J_\varepsilon^a(\mu) > 0\} \\ &\ge \frac{|\mathbb{E}\{J_\varepsilon^a(\mu)\}|^2}{\mathbb{E}\{|J_\varepsilon^a(\mu)|^2\}} \\ &\ge \frac{c}{\mathsf{Energy}_{\frac{1}{2}}(\mu)}. \end{aligned}$$

Since this holds for all $\mu \in \mathcal{P}(E)$, we obtain the desired lower bound.

Let $\{\mathcal{F}_t\}_{t \ge 0}$ denote the filtration of B and consider the martingale

$$M_t^{a,\varepsilon}(\mu) = \mathbb{E}\{J_\varepsilon^a(\mu) \,|\, \mathcal{F}_t\}, \qquad \forall t \ge 0.$$

Clearly,

$$\begin{aligned} M_t^{a,\varepsilon}(\mu) &\ge (2\varepsilon)^{-1} \int_{s \ge t} \mathbb{P}\{|B_s - a| \le \varepsilon \,|\, \mathcal{F}_t\} \, \mu(ds) \cdot \mathbf{1}_{\{|B_t - a| \le \frac{\varepsilon}{2}\}} \\ &\ge (2\varepsilon)^{-1} \int_{s \ge t} \mathbb{P}\{|B_{s-t}| \le \tfrac{1}{2}\varepsilon\} \, \mu(ds) \cdot \mathbf{1}_{\{|B_t - a| \le \frac{\varepsilon}{2}\}} \\ &\ge c\varepsilon^{-1} \int_{s \ge t} \left(\frac{\varepsilon}{\sqrt{s-t}} \wedge 1 \right) \mu(ds) \cdot \mathbf{1}_{\{|B_t - a| \le \frac{\varepsilon}{2}\}} \end{aligned}$$

Let $\sigma_\varepsilon = \inf\{s \in E : \ |B_s - a| \le \frac{1}{2}\varepsilon\}$. This is a stopping time and on $\{\sigma_\varepsilon < \infty\}$,

$$M_{\sigma_\varepsilon}^{a,\varepsilon}(\mu) \ge c\varepsilon^{-1} \int_{s \ge \sigma_\varepsilon} \left[\frac{\varepsilon}{\sqrt{s - \sigma_\varepsilon}} \wedge 1 \right] \mu(ds),$$

since all bounded Brownian martingales are continuous. Now, we choose μ carefully: WLOG $\mathbb{P}\{\sigma_\varepsilon < \infty\} > 0$ which implies that $\mu_\varepsilon \in \mathcal{P}(E)$, where

$$\mu_\varepsilon(\bullet) = \mathbb{P}\{\sigma_\varepsilon \in \bullet \,|\, \sigma_\varepsilon < \infty\}.$$

Thus, by the optional stopping theorem,

$$\begin{aligned}
1 &\geq \mathbb{E}\{M^{a,\varepsilon}_{\sigma_\varepsilon}(\mu_\varepsilon); \sigma_\varepsilon < \infty\} \\
&\geq c\varepsilon^{-1} \iint\limits_{s\geq t} \Big(\frac{\varepsilon}{\sqrt{s-t}} \wedge 1\Big)\, \mu_\varepsilon(ds)\, \mu_\varepsilon(dt) \cdot \mathbb{P}\{\sigma_\varepsilon < \infty\} \\
&\geq \frac{c}{2}\varepsilon^{-1} \iint \Big(\frac{\varepsilon}{\sqrt{s-t}} \wedge 1\Big)\, \mu_\varepsilon(ds)\, \mu_\varepsilon(dt) \cdot \mathbb{P}\{\sigma_\varepsilon < \infty\} \\
&= \frac{c}{2} \iint \Big(\frac{1}{\sqrt{s-t}} \wedge \frac{1}{\varepsilon}\Big)\, \mu_\varepsilon(ds)\, \mu_\varepsilon(dt) \cdot \mathbb{P}\{\sigma_\varepsilon < \infty\}.
\end{aligned}$$

Fix $\delta_0 > 0$ and from the above deduce that for all ε small,

$$1 \geq \frac{c}{2} \iint\limits_{|s-t|\geq\delta_0} |s-t|^{-\frac{1}{2}}\, \mu_\varepsilon(ds)\, \mu_\varepsilon(dt) \cdot \mathbb{P}\{\inf_{t\in E} |B_t - s| \leq \tfrac{1}{2}\varepsilon\}.$$

Let $\varepsilon \to 0$, and envoke Prohorov's theorem to get $\mu \in \mathcal{P}(E)$ such that

$$\mathbb{P}\{a \in B(E)\} \leq \frac{2}{c}\Big[\iint\limits_{|s-t|\geq\delta_0} |s-t|^{-\frac{1}{2}}\, \mu(ds)\, \mu(dt)\Big]^{-1}.$$

Let $\delta_0 \downarrow 0$ to finish. □

To prove the general result for a Brownian sheet, one needs the properties of the process B around a time point t. There are 2^N different notions of 'around', one for each quadrant centered at t, and this leads to 2^N different N-parameter martingales, each of which is a martingale with respect to a commuting filtration, but each filtration is indeed a filtration with respect to a different partial order. The details are complicated enough for $N = 2$ and can be found in my paper in the *Transactions of the AMS* (1999). When $N > 2$, the details are more complicated still and will be written up in the future. The end result is the following:

Theorem 4.4 (Kh and Xiao '01) *If B denotes (N, d) Brownian sheet and $E \subset \mathbb{R}^N_+$ is compact,*

$$\mathbb{E}\{|B(E)|\} > 0 \iff \mathsf{Cap}_{\frac{d}{2}}(E) > 0.$$

To recapitulate the picture we have, by Theorem 1.2, when $d \geq 2N$, $\mathbb{E}\{|B(\mathbb{R}^N_+)|\} = 0$. Thus, the above addresses portions of the range in the remaining low-dimensional case $d < 2N$. A consequence of this development is that

$$\begin{aligned}
\dim(E) > \tfrac{d}{2} &\implies |B(E)| > 0, \text{ with positive probability} \\
\dim(E) < \tfrac{d}{2} &\implies |B(E)| = 0, \text{ a.s.}
\end{aligned}$$

I will end with a related

CONJECTURE: Suppose X is an (N,d) symmetric stable sheet with index $\alpha \in (0,2)$ (see below.) Then, for any compact $E \subset \mathbb{R}^N_+$, $\mathbb{E}\{|X(E)|\} > 0$ iff $\mathsf{Cap}_{\frac{d}{\alpha}}(E) > 0$.

At the moment, this seems entirely out of the reach of the existing theory, but the analogous result for additive stable processes, and much more, holds (joint work with Xiao–will write up later.)

To finish: $\{X_t;\, t \in \mathbb{R}^N_+\}$ is an (N,d) symmetric stable sheet if it has i.i.d. cooridinates and the first coordinate has the representation $X^1_t = \int \mathbf{1}_{\{0 \preccurlyeq s \preccurlyeq t\}} \mathbb{X}(ds)$, where $\mathbb{X}$ is a totally scattered random measure such that for every nonrandom measurable $A \subset \mathbb{R}^N_+$, $\mathbb{E}\{\exp[i\xi\mathbb{X}(A)]\} = \exp(-\frac{1}{2}|A|\ \|\xi\|^\alpha)$. (Scattered means that for nonrandom measurable A and A' in $\mathbb{R}^N_+$, if $A \cap A' = \varnothing$, $\mathbb{X}(A)$ and $\mathbb{X}(A')$ are independent.)

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