MATH 2270

Final Exam - Fall 2008

Name: ______________________________

1. (18 points) Let $A = \begin{pmatrix} 1 & -2 & -10 \\ -2 & 2 & 12 \\ 4 & 4 & 8 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -1 \\ -2 \\ 20 \end{pmatrix}$.

   (a) Solve the linear system $A\vec{x} = \vec{b}$ for $\vec{x}$ using Gauss-Jordan elimination. Clearly show each step and indicate your answer. If the system is inconsistent, write *inconsistent*.

(b) What is the reduced row echelon form of A?

(c) What is rank(A)?

(d) Is the matrix A invertible?

(e) Find a basis for the kernel of A.

(f) What is the nullity of A?

(g) Find a basis for the image of A.

2. (8 points) Consider the 3×3 matrix $A = \begin{pmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ | & | & | \end{pmatrix}$ and suppose $A^T A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

(a) Find $\|\vec{v}_2\|$.

(b) Find $\vec{v}_1 \cdot \vec{v}_3$.

(c) Which (if any) of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are orthogonal?

(d) What are the possible values of $\det(A)$?

3. (10 points) Let V be the vector space spanned by the polynomials $\mathfrak{B} = \{1, 2x, 3x^2\}$ and let $T : V \to V$ be the linear transformation given by $T(f) = f'' - f' + f$.

   (a) Find the matrix of T with respect to the basis $\mathfrak{B}$.

   (b) Is T injective?

   (c) Is T surjective?

   (d) Suppose now $T(f) = f' - f''$. Find a basis for the kernel of T.

4. (8 points) Find the QR factorization of the matrix

$$M = \begin{pmatrix} 1 & 4 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Clearly show each step.

5. (10 points) Let $V = \mathbb{R}^{2\times 2}$ be an inner product space with inner product $\langle C, D\rangle = \text{trace}\left(C^T D\right)$ and let

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(a) Find the norm of A in V.

(b) Find the orthogonal projection of $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ onto the subspace of V spanned by A.

(c) Find a nonzero matrix B in V such that $\langle B, A \rangle = 0$.

(d) Suppose S is an orthogonal matrix. Prove $\langle SA, SA \rangle = \langle A, A \rangle$.

6. (10 points) Suppose

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find a diagonal matrix D and an orthogonal matrix S such that $D = S^T AS$. Show your work.

7. (4 points) For $\alpha, \beta, \gamma \in \mathbb{R}$, let

$$A = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 2 & \beta & 0 \\ 0 & 0 & 2 & \gamma \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(a) What are the eigenvalues (with multiplicities) of A?

(b) For which values of α, β, and γ is the matrix A diagonalizable?

8. (12 points) Multiple choice.

(a) The vectors $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ \alpha \end{pmatrix}$ form a basis of $\mathbb{R}^3$ for all values of α except

i. $\alpha = -2$.

ii. $\alpha = -1$.

iii. $\alpha = 0$.

iv. $\alpha = 1$.

v. $\alpha = 2$.

(b) For the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, what is the value of $\text{rank}(A) - \det(A)$?

i. -2.

ii. -1.

iii. 0.

iv. 1.

v. 2.

(c) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$T\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

will also satisfy $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$

i. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

ii. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

iii. $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

iv. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

v. Cannot be determined from the given information.

(d) Recall a matrix A is said to be *skew-symmetric* if $A^T = -A$. What is the dimension of the space of all 4×4 skew-symmetric matrices?

i. 4.

ii. 6.

iii. 8.

iv. 16.

v. None of the above.